THIN MATRIX GROUPS AND THE MONODROMY OF THE HYPERGEOMETRIC EQUATION.

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$$\Gamma \leq GL_n(\mathbb{Z})$$

$G = Zcl(\Gamma)$ Zariski closure
$\mathbb{Q}$ - algebraic group

so $\Gamma \leq G(\mathbb{Z})$.

We say Γ is arithmetic if it is finite index in $G(\mathbb{Z})$ and <u>thin</u> if not.

Many diophantine problems, standard and more exotic are connected with orbits of such a Γ: Fix $v \in \mathbb{Z}^n$,

$$\mathcal{O} := \Gamma . v \subset \mathbb{Z}^n$$

Examples of problems:

(i) If $f \in \mathbb{Z}[x_1, \ldots, x_n]$ what values does f assume on $\mathcal{O}$? Is there a local to global principle?

(ii) Can one find 'many' x's in $\mathcal{O}$ at which $f(x)$ is prime or at least has few prime factors? ("Affine Sieve").

- In the case Γ is arithmetic these are classical (and can be very difficult) problems.
- In the case that Γ is thin the problem is much more challenging but we now have the rudiments of a theory.

A key ingredient is a weak form of the Ramanujan Conjectures for such thin Γ's. These are given in terms of properties of the corresponding congruence graphs:

Fix generators $s_1, s_2, \ldots, s_t$ of Γ

$$S = \{ s_1, s_1^{-1}, \ldots, s_t, s_t^{-1} \}$$

$|S| = 2t$.

For $q \geq 1$

$$\Gamma(q) \longrightarrow \Gamma \xrightarrow[\text{mod } q]{\text{reduction}} GL_n(\mathbb{Z}/q\mathbb{Z})$$

$(\Gamma/\Gamma(q), S)$ finite Cayley graphs

$\cancel{E}\ \gamma\Gamma(q) \sim s\gamma\Gamma(q)$
$s \in S$.

Do those $|S|$ regular graphs form an <u>expander family</u> as $q \to \infty$?

Thanks to the work of many people [S-XUE], [GAMBURD], [HELFGOTT], [BOURGAIN-GAMBURD], [BOURGAIN-GAMBURD-S], [PYBER-SZABO], [BRVILLARD & GREEN-TAO], [VARJU] we have

FUNDAMENTAL EXPANSION THEOREM (SALEHI-VARJU)

$(\Gamma/\Gamma(q), S)$ IS AN EXPANDER FAMILY IFF G^0 THE IDENTITY COMPONENT OF $G := Zcl(\Gamma)$, IS PERFECT (IE. $[G^0 : G^0] = G^0$).

Application: Affine Sieve

If $f \in \mathbb{Z}[x_1, \ldots, x_n]$ and $\mathcal{O} = \Gamma v$ we say that $(\mathcal{O}, f)$ saturates if there is an $r < \infty$ such that

$\{x \in \mathcal{O} : f(x)$ has at most r-prime factors$\}$

is Zariski dense in $Zcl(\mathcal{O})$.

- The minimal such r is the saturation number $r_0(\mathcal{O}, f)$.

Example: (1) $\mathcal{O} = \mathbb{Z}$, $f(x) = x(x+2)$

$r_0(\mathcal{O}, f) = 2 \iff$ twin prime conjecture.

and

(2) THEOREM Y. ZHANG (yesterday).

$r_0(\mathcal{O}, x(x+k)) = 2$ for at least one even k less than $7 \cdot 10^7$.

FUNDAMENTAL SATURATION THEOREM AFFINE SIEVE

SALEHI-S (2013):

Γ, f as above, $\mathcal{O} = \Gamma v \subset \mathbb{Z}^n$.
If $G = Zcl(\Gamma)$ is Levi semisimple (i.e. rad G contains no torus) then
$r_0(\mathcal{O}, f) < \infty$.

· Heuristic arguments show that the condition on the radical of G is probably necessary for saturation.

For examples of local to global principles for Apollonian packings see the recent BAMS papers of FUCHS and KONTOROVICH.

<u>UBIQUITY</u> OF <u>THIN GROUPS</u> ?

• THERE IS NO DECISION PROCEDURE TO TELL WHETHER A GIVEN $A_1, A_2, \ldots, A_\ell$ $(\ell \geq 7)$ IN $SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z})$ GENERATE A THIN GROUP OR NOT (MIHALOVA 1959).

• IN PRACTICE IF Γ IS IN FACT A CONGRUENCE SUBGROUP OF $G(\mathbb{Z})$, AND IS GIVEN IN TERMS OF GENERATORS, THEN ONE CAN VERIFY THIS BY PRODUCING GENERATORS OF THE CONGRUENCE SUBGROUP.
HOWEVER IF Γ IS THIN — HOW DO WE CERTIFY THIS ?

• FOR A TRUE GROUP THEOREST THIN IS THE RULE! GIVEN $A, B \in SL_n(\mathbb{Z})$ CHOSEN AT RANDOM (SAY $\|A\|, \|B\| \leq X$ UNIFORM MEASURE) THEN WITH PROB TENDING TO ONE, $\Gamma = \langle A, B \rangle$ HAS $G = Zcl(\Gamma) = SL_n$, Γ IS FREE AND THIN. (AOUN, FUCHS)

HYPERBOLIC REFLECTION GROUPS (VINBERG):

$f(x_1, x_2, \ldots, x_n)$ a rational quadratic form of signature $(n-1, 1)$ $(n \geqslant 5)$.

$G = O_f$, $G(\mathbb{Z})$ arithmetic.

Let $R_f(\mathbb{Z})$ be the (normal) subgroup of $G(\mathbb{Z})$ generated by all $\beta \in G(\mathbb{Z})$ which induce hyperbolic reflections on $\mathbb{H}^{n-1}$, Then except for finitely many special f's

$$|O_f(\mathbb{Z}) / R_f(\mathbb{Z})| = \infty.$$

MONODROMY GROUPS: A natural geometric source of finitely generated subgroups of $GL_n(\mathbb{Z})$ is the monodromy representation on cohomology of a family of algebraic varieties, variations of hodge structures, monodromy of linear differential equations, ..

- THE FUNDAMENTAL QUESTION AS TO WHETHER IN THE CASE OF VARIATION OF HODGE STRUCTURES THE MONODROMY Γ IS ARITHMETIC WAS POSED IN 1973 BY GRIFFITHS / SCHMID.

- THEY SHOW THAT IF THE PERIOD MAP FROM THE PARAMATER SPACE S TO THE PERIOD DOMAIN D IS OPEN THEN Γ IS ARITHMETIC.

MCMULLEN (2012) CONSIDERS CYCLIC COVERS OF $\mathbb{P}^1$;

$$C_a : y^d = (x-a_1)(x-a_2)\dots(x-a_{n+1})$$

THE FUNDAMENTAL GROUP OF THE PARAMETER SPACE OF a's IS THE (PURE) BRAID GROUP.

- ANSWERING A QUESTION OF MCMULLEN VENKATAMARANA (~~2012~~ 2013) SHOWS THAT IF $n \geq 2d$ THE MONODROMY GROUP IN $GL(H_1(C)[\mathbb{Z}])$, C A BASE CURVE, IS ARITHMETIC!

- IF $n=3$ and $d=18$, Mcmullen shows that the Monodromy is thin using a relation to nonarithmetic lattices of Deligne MOSTOW.

ONE PARAMETER HYPERGEOMETRIC ${}_nF_{n-1}$:

$\alpha, \beta \in \mathbb{Q}^n$, $0 \le \alpha_j < 1$, $0 \le \beta_k < 1$

(*) — $Du = 0$, $\theta = z\frac{d}{dz}$

$$D = (\theta+\beta_1-1)\cdots(\theta+\beta_n-1) - z(\theta+\alpha_1)\cdots(\theta+\alpha_n)$$

solutions are

$$z^{1-\beta_i}\, {}_nF_{n-1}(1+\alpha_1-\beta_i, \dots, 1-\alpha_n-\beta_i ; 1+\beta_1-\beta_i, \dots, \overset{\vee}{,} \dots 1+\beta_n-\beta_i \,|\, z)$$

where $\vee$ means omit $1+\beta_i-\beta_i$ and

$${}_nF_{n-1}(\zeta_1, \dots, \zeta_n ; \eta_1, \dots, \eta_{n-1} \,|\, z) = \sum_{k=0}^{\infty} \frac{(\zeta_1)_k \cdots (\zeta_n)_k}{(\eta_1)_k \cdots (\eta_{n-1})_k} \frac{z^k}{k!}$$

(*) is singular at $0, 1, \infty$ and the monodromy group $H(\alpha, \beta)$ is gotten by analytic continuation along paths in $\mathbb{P}^1 \setminus \{0,1,\infty\}$ of a basis of solutions.

We restrict to α, β s.t. $H(\alpha, \beta)$ up to conjugation is in $GL_n(\mathbb{Z})$.

Beukers-Heckman compute

$$G = Zcl\left(H(\alpha,\beta) \right)$$

explicitly in terms of α, β.

In this self-dual setting it is one of

(i) Finite

(ii) O_n

(iii) Sp_n (only occurs if n is even)

VENKATARAMANA (2012): $n \geq 2$ even

$$\alpha = \left(\tfrac{1}{2} + \tfrac{1}{n+1}, \tfrac{1}{2} + \tfrac{2}{n+1}, \ \ldots, \tfrac{1}{2} + \tfrac{n}{n+1} \right)$$

$$\beta = \left(0, \tfrac{1}{2} + \tfrac{1}{n}, \tfrac{1}{2} + \tfrac{2}{n}, \ \ldots, \tfrac{1}{2} + \tfrac{n-1}{n} \right)$$

$G(\alpha,\beta) = Sp_n$ and $H(\alpha,\beta)$ is arithmetic!

There are 112 (α,β)'s giving $G(\alpha,\beta) = Sp_4$, __all__ come from variations of integral hodge structures.
(DORAN-MORGAN)

Of these more than half are arithmetic (Singh-Venkataramana 2012).

14 of these correspond to Calabi-Yau families of 3-folds.

eg: $(0,0,0,0)$, $(\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5})$

[PART OF DWORK FAMILY, CANDELAS ET AL MIRROR SYMMETRY FAMILY]

BRAV-THOMAS (2012) SHOW THAT THIS EXAMPLE IS THIN!

[They show that the generators of $\pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\})$ A and C about 0 and 1, play generalized ping-pong on some complicated polehedral sets in $\mathbb{P}^3$]

OF THE 14 CALABI YAU'S 7 ARE THIN AND 3 ARITHMETIC.

HYPERBOLIC HYPERGEOMETRIC (FUCHS-MEIRI-S 2013)

- (α,β) is hhm if $G(\alpha,\beta)$ is orthogonal and of signature $(n-1,1)$.

$\Rightarrow n$ must be odd.

THEOREM 1 (F-M-S)

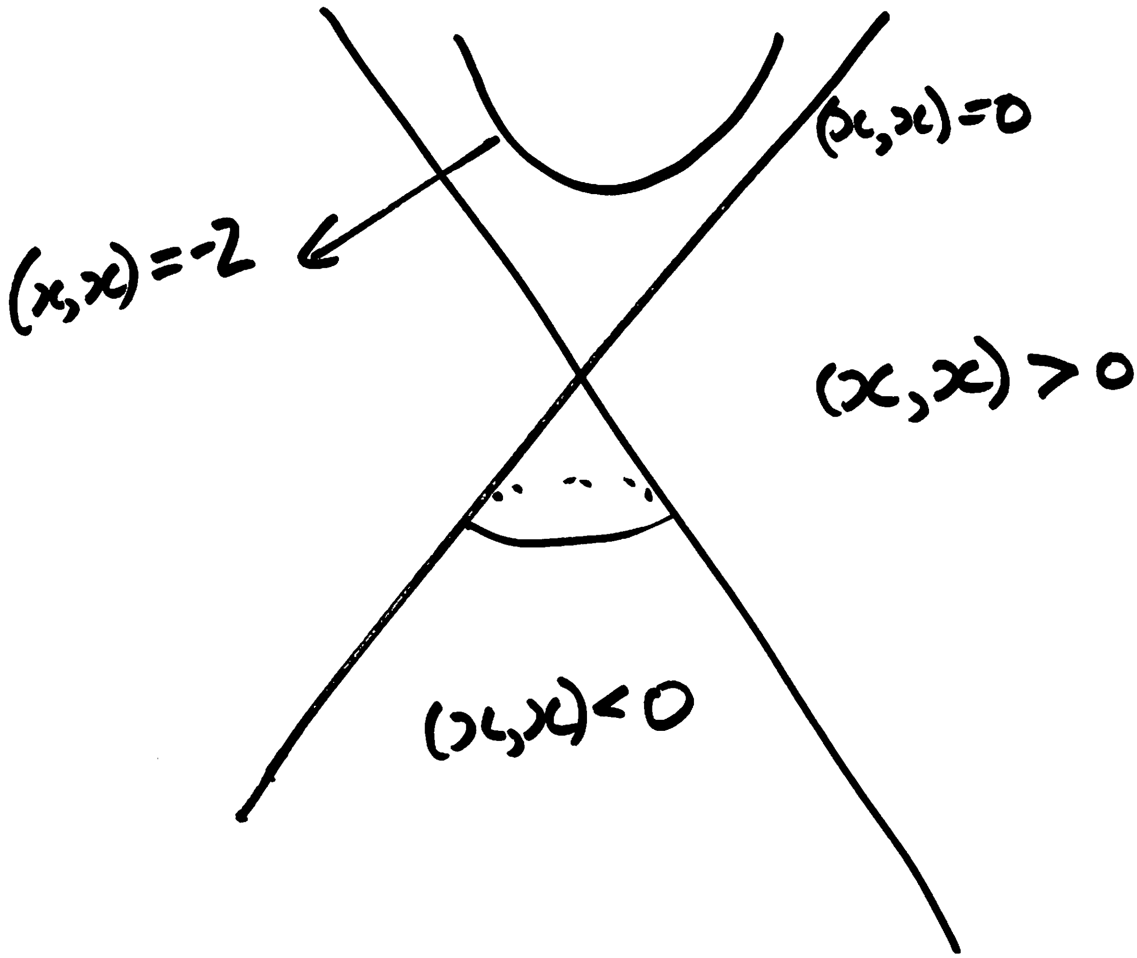
With the exception of an explicit (long) list of finitely many (α,β) (all with $n \leq 9$), all hhm's ~~~~ come in 7 parametric families.

For the hhm we give a robust obstruction for $H(\alpha,\beta)$ to be arithmetic — that is for $H(\alpha,\beta)$ to be thin.

f rational quadratic form

$f(x) = (x, x)$ integral on the lattice L.

$(n-1, 1)$



$\{(x,x) = -2 \;:\; x_1 > 0\} = \mathbb{H}^{n-1}$ model for hyperbolic space.

If $(v,v) \neq 0$ then the linear reflection

$$\tau_v(y) = y - \frac{2(v,y)}{(v,v)} v$$

is in $O(L)$ if $(v,v) = \pm 2$.

- If $(v,v) > 0$ then T_v induces a hyperbolic reflection on $\mathbb{H}^{n-1}$
- If $(v,v) < 0$ then $T_v \in O_f$ induces a <u>Cartan</u> involution on $\mathbb{H}^{n-1}$.

Key observation 1: (hhm)

$$H(\alpha,\beta) = \langle A, B \rangle$$

local monodromy A about 0, B about ∞

then $C = A^{-1}B$ is a CARTAN involution!

Up to commensurability $H(\alpha,\beta)$ is generated by Cartan involutions.

Let
$R_2(L) = \{v \in L : (v,v) = 2\}$
be the root vectors giving hyperbolic reflections.
$R_{-2}(L) = \{v \in L : (v,v) = -2\}$
the root vectors giving Cartan involutions.

According to Vinberg / Nikulin except for special f's $|O(L)/R_2(L)| = \infty$.

Let $\Delta \subset R_{-2}(L)$ we give a condition under which
$\langle \tau_v : v \in \Delta \rangle$ has finite image in $O(L)/R_2(L)$.

MINIMAL DISTANCE GRAPH $X(L)$

The points of $X(L)$ are the Cartan roots $R_{-2}(L)$

join v to w if $(v,w) = -3$.

(minimal distance these can be!)

<u>Lemma:</u> If Δ is in a connected component of $X(L)$ then $\langle T_v : v \in \Delta \rangle$ has finite image in $O(L)/R_2(L)$.

This gives the obstruction to being arithmetic. Using it we have

<u>THEOREM:</u> n odd.

$$\alpha = \left(0, \tfrac{1}{n+1}, \tfrac{2}{n+1}, \dots, \tfrac{n-1}{2(n+1)}, \tfrac{n+3}{2(n+1)}, \dots, \tfrac{n}{n+1}\right), \beta = \left(\tfrac{1}{2}, \tfrac{1}{n}, \tfrac{2}{n}, \dots, \tfrac{n-1}{n}\right)$$

and

$$\alpha = \left(\tfrac{1}{2}, \tfrac{1}{2n-2}, \tfrac{3}{2n-2}, \dots, \tfrac{2n-3}{2n-2}\right), \beta = \left(0,0,0, \tfrac{1}{n-2}, \tfrac{2}{n-2}, \dots, \tfrac{n-3}{n-2}\right)$$

are hyperbolic hypergeometrics and are arithmetic if $n=3$ and thin if $n \geqslant 5$.