April. 11. 2006. Tuesday (2:30 PM – 4:00 PM)

Christophe Breuil – 1st lecture.

4 aspects on the $\left(\begin{array}{l}\text{p-adic}\\ \text{mod } p\end{array}\right.$ local Langlands Program

$[L:\mathbb{Q}_p] < +\infty$, $d \geq 1$. ($d=0$ established)

$$\left\{\begin{array}{l}\text{p-adic Banach space}\\ (\overline{\mathbb{F}}_p\text{-v. space})\\ \text{rep'n of } GL_{d+1}(L)\end{array}\right\} \overset{?}{\longleftarrow} \left\{\begin{array}{l}(d+1)\text{-dim'l p-adic}\\ \qquad\qquad (\text{mod } p)\\ \text{rep'n of } Gal(\overline{\mathbb{Q}}_p/L)\end{array}\right\}$$

Aspect I:

When does an (irreducible) locally algebraic rep'n of $GL_{d+1}(L)$ admit at least an invariant norm?

$\|gv\| = \|v\|$ $g \in GL_{d+1}(L)$

(with Peter Schneider)

Aspect II:

(φ, Γ)-modules, $GL_2(\mathbb{Q}_p)$ (action of Borel know it's non-zero.)

Aspect III:

Drinfeld spaces, $GL_2(\mathbb{Q}_p)$ (full action of GL_2, don't know it's non-zero.)

Aspect IV:

Mod p rep'ns. $GL_2(\mathbb{Q}_{p^2})$, $GL_2(L)$

Aspect 1

$[K:\mathbb{Q}_p] < \infty$ coeff. field.

$[L:\mathbb{Q}_p] = |\mathrm{Hom}_{\mathbb{Q}_p}(L, K)|$

q = # of residue field of L.

$|x|_L := q^{-\mathrm{val}_L(x)}$, $\mathrm{val}_L(\pi_L) = 1$. π_L : uniformizer.

① Fontaine type categories

② Local Langlands revisited

③ Conjectures

④ Special Cases.

① Fontaine type categories Weil – Deligne rep'ns

$\downarrow$

" filtered (φ, N)–module without filtration"

L' a finite Galois ext'n of L.

L'_0 max unram subfield $\subseteq L'$, $\mathbb{F}_{p^{f'}}$

Important assumption

$$[L'_0 : \mathbb{Q}_p] = |\mathrm{Hom}_{\mathbb{Q}_p}(L'_0, K)|$$

$WD_{L'/L}$ = Category of rep'n (r, N, V) of the W–D gp of L on a fin. dim K–vector space V s.t. $r|_{W(\overline{\mathbb{Q}}_p/L')}$ is unramified.

$W(\overline{\mathbb{Q}}_p/L)$
$= \{ \omega \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/L) \mid \omega \mapsto (\text{abs. arith Frob in } \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p))^{d(\omega)} \}$

$r : W(\overline{\mathbb{Q}}_p/L) \longrightarrow \mathrm{Aut}_K(V)$ with open kernel + nilpotent linear endom. $N : V \to V$ s.t. $r(\omega) N r(\omega)^{-1} = p^{d(\omega)} N$

$\mathrm{MOD}_{L'/L}$:= Category of $(\varphi, N, \mathrm{Gal}(L'/L), D)$

where D = free $L'_0 \otimes_{\mathbb{Q}_p} K$ - module of finite rk.

$\varphi : D \to D$ Frobenius semi-linear on L'_0 bijective linear on K.

$N : D \to D$ linear. $N\varphi = \varphi N$ ($\Rightarrow$ nilpotent)

$\mathrm{Gal}(L'/L) \underset{\text{acts on}}{\circlearrowright} D$ (semi-linear on L'_0, linear on K) commuting with φ and N.

Fix $\sigma'_0 : L'_0 \hookrightarrow K$ (from the assumption)

<u>Fontaine</u>

$$\mathrm{WD} : \mathrm{MOD}_{L'/L} \longrightarrow \mathrm{WD}_{L'/L}$$

$$(\varphi, \underset{N_D}{\overset{}{N}}, \mathrm{Gal}(L'/L), D) \longmapsto (r, N, V)$$

$$D \longmapsto V := D \otimes_{L'_0 \otimes_{\mathbb{Q}_p} K,\ \sigma'_0 \otimes \mathrm{Id}} K$$

$$N : V \to V. \quad N := N_D \otimes \mathrm{Id}$$

$$r(\omega) : V \to V. \quad r(\omega) := \bar{\omega} \circ \varphi^{-d(\omega)}$$

$$\bar{\omega} \in \mathrm{Gal}(L'/L)$$

<u>Lemma:</u> <u>WD</u> is an equivalence of categories.

$$D \simeq \bigoplus_{n=0}^{f'-1} \underbrace{V_{\sigma'_0 \circ \varphi_0'^{-n}}}_{= D \otimes_{L'_0 \otimes_{\mathbb{Q}_p} K,\ \sigma'_0 \circ \varphi_0'^{-n} \otimes \mathrm{Id}} K}$$

where φ'_0 = Frob on L'_0. $\quad D = \bigoplus_{n=0}^{f'-1} V$

② Local Langlands Revisited.

$$\left\{ \begin{array}{l} \text{isom. classes of } \underline{\text{smooth}} \\ \underline{\text{irred.}} \text{ rep'n of } GL_{d+1}(L)/\overline{\mathbb{Q}}_p \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{isom classes of WD} \\ \text{rep'n } (r, N, V) \ / \overline{\mathbb{Q}}_p \end{array} \right\}$$

r : semi-simple

normalized reciprocity map

$$\mathrm{rec} : W(\overline{\mathbb{Q}}_p/L)^{ab} \xrightarrow{\sim} L^\times$$

arith. Frob $\longmapsto$ inverse of uniformizer

$$(r, N, V) \longmapsto \pi^u \text{ s.t. } (\text{Central char})(\pi^u) = \det(r, N, V) \circ \mathrm{rec}^{-1}$$

π^u depends on $\sqrt{q}$.

$$(r, N, V) = \underbrace{\bigoplus_{i=1}^{s} (r_i, N_i, V_i)}_{\text{indecomposable}} \ / \overline{\mathbb{Q}}_p$$

indecomposable (nontrivial monodromy operator)

$$\underbrace{\pi_i^u}_{} \longleftrightarrow (r_i, N_i, V_i)$$

Generalized Steinberg rep'n (can include super-cuspidal ones irred (r_i, N_i, V_i))

$$\mathrm{Ind}_P^{GL_{d+1}} (\pi_1^u \otimes \quad \otimes \pi_s^u) \twoheadrightarrow \pi^u$$

(can be reducible)

Normalized Parabolic Induction

$$\pi \overset{\mathrm{def}}{=} \left(\mathrm{Ind}_P^{GL_{d+1}} (\pi_1^u \otimes \quad \otimes \pi_s^u) \right) \underset{\overline{\mathbb{Q}}_p}{\otimes} |\det|_L^{-\frac{d}{2}}$$

$\boxed{\text{Lemma}}$ ← Bernstein-Zelevinsky Thy

Assume (r, N, v) is a rep'n on fin dim K-v. space.
Then π admits a unique model over K.
Moreover π doesn't depend anymore on $\sqrt{q}$.

Ex $d = 1$

$$\Pi^{\text{unit}} = |\cdot|_L \longleftrightarrow (r, N, V) = \begin{pmatrix} |\cdot|_L & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Pi = \mathrm{Ind}_{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}}^{GL_2} |\cdot|_L \otimes |\cdot|_L^{-1} \qquad (\text{not normalized})$$

③ Conjecture

Fix (r, N, V), r: semi-simple $\rightsquigarrow \Pi$

- for each $\sigma : L \hookrightarrow K$, integers $i_{j,\sigma} \in \mathbb{Z}$ such that

$$i_{1,\sigma} < i_{2,\sigma} < \cdots < i_{d+1,\sigma} \quad (\text{Strict Inequalities})$$

(opposite of Hodge-Tate weights)

Define ρ_σ = K-rational alg. rep'n of $GL_{d+1}(K)$ of highest wt $-i_{d+1,\sigma} \le -i_{d,\sigma^{-1}} \le \le -i_{1,\sigma^{-d}}$

$$\rho_\sigma = \left(\mathrm{Ind}_{\begin{pmatrix} x_1 & * \\ 0 & x_{d+1} \end{pmatrix}}^{GL_{d+1}(K)} x_1^{-i_{d+1,\sigma}} \otimes x_2^{-i_{d,\sigma^{-1}}} \otimes \cdots \otimes x_{d+1}^{-i_{1,\sigma^{-d}}} \right) \subseteq H^0(GL_{d+1}, \mathcal{O}_{GL_{d+1}})$$

$\rho \underset{\sigma}{\otimes} \rho_\sigma$ = rep'n of $GL_{d+1}(L)$ acting diagonally.

For σ. $GL_{d+1}(L) \hookrightarrow GL_{d+1}(K)$ via $\sigma : L \hookrightarrow K$.

$\boxed{\text{Conjecture}}$: The following conditions are equivalent.

(i) There is an invariant norm on $\rho \underset{K}{\otimes} \Pi$

(ii) There is an object $(\varphi, N, \mathrm{Gal}(L'/L), D) \in \mathrm{MOD}_{L'/L}$ such that $\mathrm{WD}(\varphi, N, \mathrm{Gal}(L'/L), D)^{F\text{-ss}} \simeq (r, N, V)$ and a (weakly) admissible filtration preserved by $\mathrm{Gal}(L'/L)$ on $D_{L'} := L' \underset{L_0'}{\otimes} D$

s.t. $(*)$ $\dfrac{\mathrm{Fil}^i D_{L',\sigma}}{\mathrm{Fil}^{i+1} D_{L',\sigma}} \neq 0 \iff i \in \{ i_{1,\sigma}, \dots, i_{d+1,\sigma} \}$ $\quad \overbrace{\simeq \prod_{\sigma : L \hookrightarrow K} D_{L'} \underset{L' \otimes_{\mathbb{Q}_p} K}{\otimes} (L' \underset{L,\sigma}{\otimes} K)}^{D_{L',\sigma}}$

Weak Conjecture

Prop The central character of $\rho \otimes_K \pi$ is integral iff for any filtration satisfying (*) one has $t_H(D_{L'}) = t_N(D)$

Proof. Central char. of $\rho \otimes_K \pi$ is integral iff $\mathrm{val}((\text{central char of } \rho)(\pi_L)) + \mathrm{val}((\text{cen. char. of } \pi)(\pi_L)) = 0$

$$\mathrm{val}_L(\text{c. char } \rho\,(\pi_L)) = -\sum_{j=1}^{d+1} \sum_{\sigma: L \hookrightarrow K} (i_{d+2-j,\sigma} + (j-1))$$

$$\mathrm{val}_L(\text{c. char } \pi(\pi_L)) = -\mathrm{val}_L(\det{}_K(r)(\text{arith Frob of } W(\overline{\mathbb{Q}}_p/L))) + [L:\mathbb{Q}_p]\frac{d(d+1)}{2}$$

$\sigma_0' : L_0' \hookrightarrow K$

$$D_{\sigma_0'} := D \otimes_{L_0' \otimes_{\mathbb{Q}_p} K,\ \sigma_0' \otimes \mathrm{Id}} K$$

$$-\mathrm{val}_L(\det{}_K(r)(\text{arith Frob of } W(\overline{\mathbb{Q}}_p/L))) = \frac{f}{f'}\mathrm{val}_L(\det{}_K(\varphi^{f'} \mid D_{\sigma_0'}))$$

$$\varphi^{f'} : D_{\sigma_0'} \to D_{\sigma_0'}$$

$$\mathrm{val}_L(\text{c char } \pi(\pi_L)) = \frac{f}{f'}\cdot\mathrm{val}_L(\det{}_K(\varphi^{f'} \mid D_{\sigma_0'})) + [L:\mathbb{Q}_p]\frac{d(d+1)}{2}$$

$$t_H(D_{L'}) = \sum_{\sigma}\sum_{j=1}^{d+1} [K:L]\, i_{j,\sigma}$$

$$t_N(D) = [K:L]\,\frac{f}{f'}\,\mathrm{val}_L(\det{}_K(\varphi^{f}\mid D_{\sigma_0'}))$$

$$\Rightarrow \mathrm{val}_L(\text{c char } \rho(\pi_L)) + \mathrm{val}(\text{c char } \pi(\pi_L)) = \frac{1}{[K:L]}(-t_H(D_{L'}) + t_N(D)).$$

Cor. The Conjecture holds if r is irreducible (equally if π is super-cuspidal.)

Proof. $\pi = c\text{-ind}_{U Z}^{GL_{d+1}(L)} \lambda$ $\qquad Z = L^\times$, $U \subseteq GL_{d+1}(O_L)$, λ: fin. dim $/K$.

π has an invariant lattice

iff $\rho \otimes_K \lambda$ has one. $\rho \otimes \pi = c\text{-ind}_{U\cdot Z}^{GL_{d+1}(L)} (\rho \otimes_K \lambda)$

$\rho \otimes \pi$ has a lattice iff its central char. does.

$$\begin{array}{ccc} (\varphi, N, Gal(L'/L), D) \in MOD_{L'/L} & & (D \text{ is irreducible}) \\ \updownarrow \text{WD} & & \updownarrow \\ (r, N, V) & & \\ \therefore \; t_H = t_N & & r \text{ is irreducible} \end{array}$$

④ Special Case (Example)

Ex. $L = L' = \mathbb{Q}_p$

$d = 1$, $N = 0$

- r : unramified

arithmetic Frob. of $Gal(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \longrightarrow \begin{pmatrix} p^{\frac{k-1}{2}} & \\ & p^{\frac{k-1}{2}} \end{pmatrix}$

- $i_1 = 1 - k < i_2 = 0$ $\qquad \begin{pmatrix} |\cdot|^{\frac{k-1}{2}} & \\ & |\cdot|^{\frac{k-1}{2}} \end{pmatrix}$

$k \geq 2$

$\rho \otimes_k \pi$ has an invariant norm

$$\varphi(e_1) := p^{-\frac{k-1}{2}} \cdot e_1$$

$$\varphi(e_2) := p^{-\frac{k-1}{2}} \cdot (e_1 + e_2)$$

$$\mathrm{Fil}^{-(k-1)+1} = \cdots = \mathrm{Fil}^0 = K \cdot (e_1 + e_2) \quad \text{is admissible}$$

φ: <u>not</u> <u>semi-simple</u>

<u>Thm</u> (Schneider - Teitelbaum - B.)

Assume that (r, N, V) is a direct sum of unramified characters, then (i) ⇒ (ii) in the conjecture

Sketch of the proof:

$$r: \begin{array}{l}\text{arithmetic} \\ \text{Frob. of} \\ W(\overline{\mathbb{Q}}_p/L)\end{array} \longmapsto \begin{pmatrix} \zeta_1 & & \\ & \ddots & \\ & & \zeta_{d+1} \end{pmatrix} \in (K^\times)^{d+1}$$

$$U := GL_{d+1}(O_L)$$

$$G := GL_{d+1}(L)$$

$$\mathcal{H}(G, 1_U) := \mathrm{End}_G\left(c\text{-}\mathrm{ind}_U^G 1_U\right) = \{ f : U \backslash G / U \to K, \text{ cpt supported} \}$$

$$\mathcal{H}(G, \rho|_U) := \mathrm{End}_G\left(c\text{-}\mathrm{ind}_U^G \rho|_U\right) = \{ f : G \to \mathrm{End}_K(V_\rho),\ f(u_1 g u_2) = f(u_1) \cdot f(g) \cdot f(u_2), \text{ and cpt supported} \}$$

split torus

$T \subset G$, $T^\circ := T \cap U$

$$\hat{\zeta} : T/T^\circ \to K \qquad \hat{\zeta} := \mathrm{unr}(\zeta_1) \otimes \mathrm{unr}(\zeta_2) |\cdot|_L \otimes \cdots \otimes \mathrm{unr}(\zeta_{d+1}) |\cdot|_L^d$$

Fact : $\boxed{\Pi \simeq K \underset{\hat{\xi},\, \mathcal{H}(G, \mathbb{1}_U)}{\otimes} c\text{-}\mathrm{ind}_U^G \mathbb{1}_U}$

$$\mathcal{H}(G, \mathbb{1}_U) \hookrightarrow K[T/T^\circ] \xrightarrow{\hat{\xi}} K$$

Satake map

$$\hat{\xi}$$

$\rho^\circ :=$ U-invariant lattice inside ρ

$\longrightarrow$ invariant norm on $c\text{-}\mathrm{ind}_U^G \rho|_U$

$\longrightarrow$ ———— ———— on $\mathrm{End}_G(c\text{-}\mathrm{ind}_U^G \rho|_U)$ $\shortparallel$

So invariant norm on $\mathcal{H}(G, \rho|_U)$

$$\mathcal{B}(G, \rho|_U) = \text{Completion of } \mathcal{H}(G, \rho|_U)$$

$$\hat{\xi} : \mathcal{H}(G, \rho|_U) \xrightarrow[i^{-1}]{} \mathcal{H}(G, \mathbb{1}_U) \xrightarrow{\xi} K$$

Assume $\rho \otimes_K \Pi$ has an invariant norm. ((i) in Conjecture)

$$\shortparallel$$

$$K \underset{\hat{\xi},\, \mathcal{H}(G, \rho|_U)}{\otimes} c\text{-}\mathrm{ind}_U^G \rho|_U$$

$\Rightarrow$ The image of $c\text{-}\mathrm{ind}_U^G \rho|_U$ by $\hat{\xi}$ in K is bounded

Computation : $(\mathrm{val}_L(\xi_1), \ldots \mathrm{val}_L(\xi_{d+1}))^{\mathrm{dom}} \in \mathbb{Q}^{d+1}$

$$\leq \left(\sum_\sigma a_{1,\sigma}, \quad \sum_\sigma a_{d+1,\sigma}\right) + [L : \mathbb{Q}_p](0, 1, \ d)$$

$$a_{j,\sigma} := -i_{d+2-j\,\sigma} - (j-1)$$

$x_i \leq x_{i+1}$, $y_i \leq y_{i+1}$

$(x_1, \ldots, x_{d+1}) \leq (y_1, \ldots y_{d+1})$

$\Leftrightarrow$

$x_{d+1} \leq y_{d+1}$

$x_d + x_{d+1} \leq y_d + y_{d+1}$

$\vdots \qquad \vdots$

$x_1 + \cdots + x_{d+1} \leq y_1 + \cdots + y_{d+1}$

and equality of $t_H = t_N$ implies the existence of one admissible filtration on D. ((ii) in Conjecture)