Mar 14 2006 Tuesday. 1:00 – 2:30 p.m. Kevin Buzzard 7th (E.V.) lecture.

Let me do something properly.

Let p be a prime.

& let's consider a modular curve of level prime to p

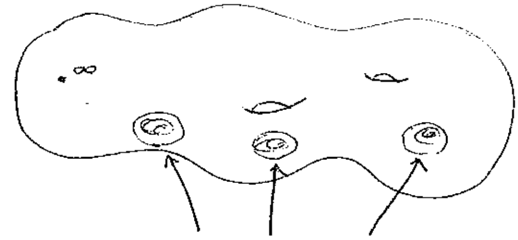
e.g. $X_1(N)$ or $X(N)$. $p \nmid N$.

If the level is "sufficiently small", eg. $X_1(N)$, $N \geq 4$, or $X(N)$ $N \geq 3$

then the non-cuspidal pts of $X_1(N)$ represent a functor of elliptic curves (over a rather general base – but let's

Assume for the moment that this is the case

$X_1(N)\ /\mathbb{Q}_p$



$\overline{\mathbb{F}}_p$

discs correspond to ell. curves with good s.s. reduction.

Choose a center of each disc defined over $\mathbb{Q}_p^{nr}$. the max unram. ext'n of $\mathbb{Q}_p$.

Choose an isomorphism $\quad j \mapsto j+p^2$

$$\text{s.s. disc} \longrightarrow \text{open unit disc}$$
$$\text{center} \longmapsto 0$$

is isom. defined over $\mathbb{Q}_p^{nr}$

Then all other choices of centre in s.s. disc correspond to pts in $\left(\begin{array}{c}\text{closed unit disc}\\ \text{radius } \frac{1}{p}\end{array}\right) \subset$ open unit disc.

So if $0 < r < 1$ it makes sense to say that a point $x \in$ s.s. disc is "distance p^{-r} from the center". $\quad 0 < r < 1$

$$1 > p^{-r} > \frac{1}{p}$$

Careful analysis of formal gps shows that if $0 < r < \frac{p}{p+1}$ then the point x corresponds to an elliptic curve with s.s. reduction but with a canonical subgp

Notation: If $0 < r < \frac{p}{p+1}$, then let $X_1(N)[r]$

$=$ subspace (affinoid subdomain) of $X_1(N)$ consisting of ord. locus & $x \in$ s.s. locus s.t

$$d(x, \text{center}) \geq p^{-r}.$$

Define $X_1(N)[0]$ = ordinary locus.

An r-overconvergent modular ftn (of level $\Gamma_1(N)$) is a rigid analytic ftn on $X_1(N)[r]$.

If level structure is not "fine" enough then add an auxiliary "Galois" level structure (e.g. full level ℓ structure) for $\ell >> 0$ not dividing anything

Set $\Gamma' = \Gamma \cap \Gamma(\ell)$

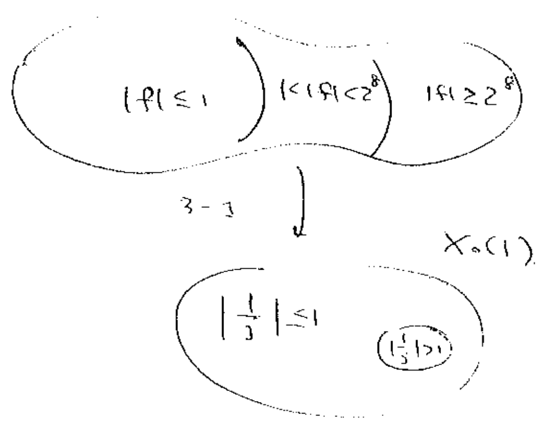
Then $X(\Gamma')[r]$ is defined as above

& $X(\Gamma)[r] = X(\Gamma')[r] / (\Gamma/\Gamma')$

Explicitly the functions on $X(\Gamma)[r]$ are Γ-invariant ftns on $X(\Gamma')[r]$ (independent of Γ')

Our toy example:

$N = 1$. $p = 2$ $X_0(2)$



It turns out that using j to give an isomorphism s.s disc $\xrightarrow{\sim}$ open unit disk is the wrong thing to do — we're out by a factor of 12.

Explicitly if $x \in X_0(1)$

& $r(x) = r \in (0, \frac{p}{p+1}) = (0, \frac{2}{3})$

(r is st dist to center is 2^r, when suff. rigid) then $|j(x)| = 2^{-12r}$

If $0 < r < \frac{p}{p+1}$)

then all points in $X_1(N)[r]$ have canonical subgp.

So if $X_1(N;p)$ is representing ell. curves
+ pt ord N
+ subgp order p

then the forgetful functor

$$X_1(N;p) \longrightarrow X_1(N)$$

has a canonical section on $X_1(N)[r]$

$E \to E$ canonical subgp

The ftn V

can be understood thus:

if $r < \frac{1}{p+1}$

then there is a map

$$X_1(N)[r] \longrightarrow X_1(N)[pr]$$
$$E \longmapsto E/C$$

C = canonical subgp.

The map is finite & flat of degree p

& the induced map on ftns

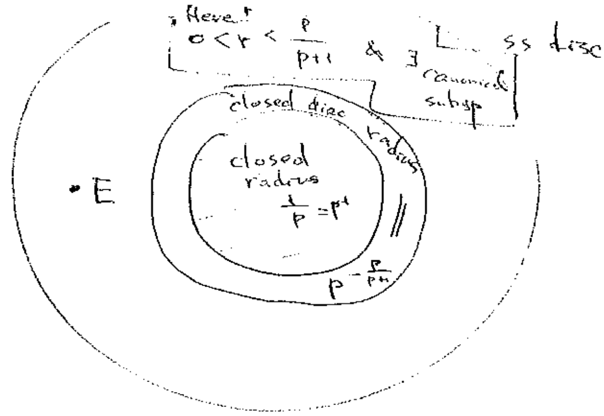
$$\mathcal{O}(X_1(N)[pr]) \longrightarrow \mathcal{O}(X_1(N)[r])$$

is $f(q) \longmapsto f(q^p)$

The operator U is a map

$$\mathcal{O}(X_1(N)[r]) \longrightarrow \mathcal{O}(X_1(N)[pr])$$

$r < \frac{1}{p+1}$



<u>Facts</u>. if $r(E) < \frac{1}{p+1}$, then $\exists\, C \subset E$ canonical

& $r(E/C) = p\, r(E)$

If $r(E) < \frac{p}{p+1}$ & C is canonical

& $D \neq C$, D order p

then $r(E/D) = \frac{r(E)}{p}$

Because U is a cts map

$$\mathcal{O}(X_1(N)[r]) \longrightarrow \mathcal{O}(X_1(N)[pr])$$

the CPS of U on r-overconvergent ftns is independent of r for $r > 0$

$$\begin{tikzcd}
\mathcal{O}(X_1(N)[r]) \arrow[r, "U"] \arrow[dr, "U_1"'] & \mathcal{O}(X_1(N)[pr]) \arrow[d, "\text{res}"] \arrow[dr, "U_2"] & \\
& \mathcal{O}(X_1(N)[r]) \arrow[r, "U"] & \mathcal{O}(X_1(N)[pr])
\end{tikzcd}$$

$$\text{CPS}(U_1) = \text{CPS}(U_2)$$

But when trying to prove things about eg. the spectral curve, one sometimes really cares about how far things overconverge.

<u>eg.</u> if $p = 2$ & $N = 1$

Kilford + Buzzard

Showed that for κ near center of $W^\circ = W^+$
the formal q-expansion E_κ/V_κ was a fn
on $X_0(1)[r]$ for any $r < \frac{1}{4}$.
$\parallel$
closed disc

This has consequences for the fn E_κ/V_κ $\quad V_\kappa = V(E_\kappa)$
for all $\kappa \in W^\circ$ $\quad = E_\kappa(q^p)$

Using these consequences, we could prove just enough about valuations of entries in matrix representing U on wt κ forms to compute the NP of CPS(U)

& we deduced that if $w \in W^\circ$, $|w| > \frac{1}{8}$
then the norms of the U-eigenvalues of the wt k eigenforms ($\kappa \longmapsto w$)
are 1, $|w|$, $|w|^2$, $|w|^3$, ...
E_κ each occuring with multiplicity 1

Funny Consequences

1) If $|w| > \frac{1}{8}$
& f is a wt k U-eigen form, then f is an eigen fcn for all the Hecke operators

2) If $\kappa(x) = x^k \chi(x)$
χ a Dirichlet char of conductor 2^n
& $\chi(-1) = (-1)^k$
then the classical wt κ level 2^n character χ eigenforms ...
are overconvergent.

f/E_κ mero fn on $X_0(2^n)[r]$
$\downarrow$
$X_0(1)[r]$

$k \geq 1$. E_k classical, wt k, level 2^n, char χ

Furthermore if $n \geq 2$

then $k \leftrightarrow w$. $w = \kappa(5)-1 = 5^k \chi(5)-1$ & $|w| > \frac{1}{8}$.

Classical results about U_p

$\Rightarrow$ if f is classical eigenform wt k level 2^n char χ as above, then $v(a_2(f)) \leq k-1$

$\therefore |a_2(f)| \geq 2^{-(k-1)}$

& One checks that the number of elements of the set

$$\{ x \in \{1, |w|, |w|^2, \cdots\}, \ |x| \geq 2^{-(k-1)} \}$$

= dim of space of classical modular forms wt k level 2^n, char χ

Hence we know val'n of all eigenvalues of U_2 on $M_k(\Gamma_1(2^n), \chi)$

cond$(\chi) = 2^n$

$n \geq 2$

Consequence 3.

$f \in M_k(\Gamma_1(2^n), \chi)$ normalized eigenform

$\Rightarrow$ q-exp$(f) \in \mathbb{Q}_2(\chi)[[q]]$.

$T_2 \curvearrowright S_k(SL_2(\mathbb{Z}))$

(3 cusps) $M_k(\Gamma_1(4))$ $\forall k \geq 1$ odd

$\underline{0}, 1, 2, 3, \ldots, \underline{k-1}$

converging

3-Eisenstein series

Consequence 4:

The Spectral curve for $p=2$, $N=1$ over $\{|w| > \frac{1}{8}\}$ is a disjoint union of annuli, each is isomorphic to $\{|w| > \frac{1}{8}\}$

How might one generalize these results to general p?

Here's an example of an ingredient one would need

Given $\kappa \in \mathcal{W}^0$, produce an $r > 0$

$E_\kappa / V(E_\kappa)$ is a fctn on $X_0(1)[r]$

$\kappa = 0$ then $E_\kappa = 1$, r can be anything $< \frac{p}{p+1}$

If $\kappa \neq 0$, then $E_\kappa / V_\kappa - 1$ is in the kernel of U.

$$U\left(E_k/V_k\right) = \frac{1}{E_k}U(E_k) = \frac{E_k}{E_k} = 1 \qquad \& \ U(1) = 1$$

& it's a general fact that U is injective on $X_1(N)\left[\frac{1}{p+1}\right]$

$U(F\,V(G)) = G \cdot U(F)$

$V(\frac{1}{t}) = \frac{1}{V(t)}$

$\therefore$ one needs $r < \frac{p}{p+1}$.

$f \in \mathcal{O}(X_1(N)[\frac{1}{p+1}]$

$Uf \in \mathcal{O}(X_1(N)[\frac{p}{p+1}] \subset X_1(N;p)$

$(Uf)(E,C) = 0$

$\boxed{\sum_{D \neq C} f(E/D) = 0}$

$\sum_D f(E/D) = 0$

$f(E/D) = 0 \ \forall D$

Remark: Calegari & I proved that the 2-adic spectral curve "had no holes"

i.e. if D is a disc & $D^* = D - pt$

& we have a map $D^* \xrightarrow{f}$ Spectral curve

s.t g extends to D

then f extends ~~too~~.

$$\begin{tikzcd} D^* \arrow[r, "f"] \arrow[d, hook] & \text{Spectral curve} \arrow[d] \\ D \arrow[r, dotted] \arrow[ur, dotted, "g"] & W \end{tikzcd}$$

$\left(T_p(E_k^{clas}) = (1+p^{k-1})E_k^{clas}\right)$

E_k: classical

$\left(U_p(E_k) = E_k\right)$

E_k: overconv

$\mathbb{E} - U(\mathbb{E}) \in \mathcal{O}(W)[[q]]$

Vague reason why a general spectral curve might be proper (p, N) is that forms in $\ker(U)$ are not very overconvergent

OTOH if f is overconvergent & $Uf = \lambda f$, $\lambda \neq 0$, then f is $\underline{\text{very}}$ overconvergent

as if f is r-overconvergent, then $f = \frac{U(f)}{\lambda}$ is pr-overconvergent $\Rightarrow f$ is r-overconv $\forall r < \frac{p}{p+1}$

I'll now show you the $(p=2)$ disappointing pf that for $k \in W^0$, $k \leftrightarrow w$, $|w| < \frac{1}{8}$

$$E_k/V_k \in \mathcal{O}(X_0(1)[r]), \ \forall r < \frac{1}{4}.$$

Q) General p. $V_k = V(E_k)$

$k \in W^\circ$

Say $E_k/V_k \in \mathcal{O}(X_0(1)[r])$

$r < \frac{1}{p+1}$.

Is it true that E_k/V_k has no zeros on $X_0(1)[r]$?

Strategy: (Emerton)

Say F is any wt $\overset{W^\circ}{\in}$ overconvergent MF.

s.t. $F \in \mathcal{O}_p[[q]]$ & $\overline{F} \in \overline{\mathbb{F}}_p[[q]]$ is 1. (e.g. $F = E_k$) $(*)$

F/E_k is an overconvergent fn.

$\therefore$ VF/V_k is overconvergent (& non-zero)

& $(*)$ $\Rightarrow$ $\exists\, r > 0$ s.t. F/E_k is non-zero on $X_0(1)[r]$

$\Rightarrow$ F/VF is an overconvergent fn.

Assume F/VF is r_0-overconvergent & ~~non-vanishing~~ 1-unit on $X_0(1)[r_0]$

Assume also that $\forall r \leq r_0$ (f: 1-unit $\Leftrightarrow |f-1| < 1$, $|f(x)-1| < 1\ \forall x \in X_0(1)[r]$ $\Rightarrow |f(x)| = 1$)

$U : \mathcal{O}(X_0(1)[r]) \longrightarrow \mathcal{O}(X_0(1)[pr])$ has norm ≤ 1

Then E_k/V_k is r_0-overconvergent & a 1-unit $(\dagger)$

Application: $F = G^k$ G explicit wt 1 Eisenstein series

$p=2$

$(\dagger)$ is true $(F = U(G^k))$

Pf) E_k/F is a 1-unit on $X_0(1)[0]$ & hence on $X_0(1)[\varepsilon]$

for some $\varepsilon > 0$.

Furthermore, if $\tilde{U}$ is the operator $g \mapsto U(g \times F/VF)$

then $\tilde{U}(E_k/F) = E_k/F$.

& $(\dagger) \Rightarrow E_k/F$ is a 1-unit on $X_0(1)[p\varepsilon]$

Keep iterating $\tilde{U}$

$\Rightarrow$ E_k/F is pr_0-overconv & a 1-unit

$\therefore$ V_k/VF is r_0-overconv. & a 1-unit

$\Rightarrow$ E_k/V_k is r_0-overconv. & a 1-unit.

Turns out that $(\dagger)$ is true for $p=2$ (Emerton)

$(\dagger)$ is False for $p = 13$ (Loeffler)

True for $p=3$

$\exists$ f: ordinary form $f \neq 1$