May 4. 2006. Thursday Kevin Buzzard (final lec)

21th lecture.

Last time:

$F/\mathbb{Q}_p$ finite

$F \supset \mathcal{O} \ni \pi.$ ($\mathcal{O}/\pi = k$)

$G = GL_n(F)$

$\cup$

$B = \begin{pmatrix} * & * & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$ $K = GL_n(\mathcal{O})$ . $Z = F^\times \subseteq G$

$c\text{-}\mathrm{Ind}_{KZ}^{G} V$ — big. rep'n of $G$.

$V$ is a f-d. mod p rep'n of $KZ$. $V/E$, $E = \overline{\mathbb{F}}_p$

$V$ = rep'n of $\Gamma = GL_n(k)$

$\uparrow$

$K$.

$\mathcal{H}(V) = \mathrm{End}(c\text{-}\mathrm{Ind}_{KZ}^{G} V)$

- probably a poly ring $n(n-1)$-variables
- true if $n = 2$

Idea: if $m \subset \mathcal{H}$ is a max. ideal

$\dfrac{c\text{-}\mathrm{Ind}_{KZ}^{G}(V)}{m}$ is a rep'n of $G$. $\boxed{\mathrm{Ind}_{B}^{G} \chi}$

Basic fact: if $W$ is any v. sp /E in which $G$ acts irred. & smoothly (w ∈ W has open stabilizer) & W has a central character (possibly with $\pi \in Z$ acting non-trivially) then $W$ is a quotient of $c\text{-}\mathrm{Ind}_{KZ}^{G} V$ for some $V$.

$$0 \longrightarrow K(1) \longrightarrow K \longrightarrow \Gamma \longrightarrow 0$$

PB Let $I(1) \subseteq K$ be things$^{\text{pro-}p}$ which reduce to

$$\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \quad \text{in} \quad GL_n(k)$$

Then $I(1)$ is pro-$p$.

& one checks easily that

if $W \neq 0$, then $W^{I(1)} \neq 0$.

(pf. reduce to case of fin. $p$-gp)

& if $\overline{T} \subseteq GL_n(k)$ is the diagonal matrices

then $\overline{T}$ acts on $W^{I(1)}$

finite ab. gp of order prime to $p$.

$W^{I(1)}$ now breaks up into $\overline{T}$-eigen spaces

$\therefore \exists \ \varepsilon : \overline{T} \longrightarrow E^\times$

& $w \in W^{I(1)}$, $w \neq 0$. $tw = \varepsilon(t) \cdot w$.

$w$ is now $\overline{B}$-stable. $\overline{B}$ - upper $\Delta$ matrices in $GL_n(k)$.

$\Gamma = GL_n(k)$.

& Frob. Rec.

$$\mathrm{Hom}\left(\mathrm{Ind}_{\overline{B}}^{\Gamma} \varepsilon, W^{K(1)}\right) \neq 0$$

$\therefore \exists \ V$, some J-H factor of $\mathrm{ind}_{\overline{B}}^{\Gamma} \varepsilon$ s.t.

$\mathrm{Hom}_{\Gamma}(V, W^{K(1)}) \neq 0$.

Let $Z$ act on $V$ via central char. of $W$

$\mathrm{Hom}\left(c\text{-}\mathrm{ind}_{KZ}^{G} V, W\right) \neq 0$.

$W$ irred. $\Rightarrow$ $W$ is a quotient of $c\text{-}\mathrm{ind}_{KZ}^{G} V$.

Now let's restrict to $n = 2$.

The irred. mod $p$ rep'ns of $\Gamma = GL_2(k)$

$E$ - rep'ns

are all twists of $\mathrm{Symm}^{\underline{r}}(E^2)$

where $\underline{r} = (r_1, r_2, \dots, r_f)$ is a vector.

$0 \le r_i \le p-1 \quad \forall i$

$\#k = p^f$

& $\mathrm{Symm}^{\underline{r}}(E^2) := \bigotimes_{i=1}^{f} \mathrm{Symm}^{r_i}(k^2 \otimes_{\sigma_i} E)$, $E = \overline{\mathbb{F}}_p$, $\circlearrowleft GL_2(k)$

where $\sigma_1, \sigma_2, \dots, \sigma_f : k \to E$

are the embeddings.

for now choose

$\underline{r} = (r_1, r_2, \dots, r_f)$, $0 \le r_i \le p-1$

$\lambda \in E$. $\eta : F^\times \to E^\times$. & define $\pi(\underline{r}, \lambda, \eta)$

to be $\dfrac{c\text{-}\mathrm{ind}_{KZ}^{G}(\mathrm{Symm}^{\underline{r}}(E^2))}{T - \lambda} \otimes \eta \circ \det$.

$T \in \mathcal{H}(V)$ supported on $KZ \begin{pmatrix} 1 & 0 \\ 0 & \varpi \end{pmatrix} KZ$.

"is" $E[T]$

Barthel - Livne analyze

$\pi(\Sigma, \lambda, \eta)$ for $\underline{\lambda \neq 0}$.

Aside principal series

$T$ = diag matrices in $G = GL_2(F)$

$\uparrow$

$B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$

If $R$ is any comm. ring.

& $\chi = (\chi_1, \chi_2) : T \longrightarrow R^\times$.

$$\chi\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \chi_1(a) \cdot \chi_2(d).$$

then we can form un-normalized induction

$$\mathrm{Ind}_B^G \chi = \{ \text{ftnc } f : G \longrightarrow R \text{ s.t. } f(bg) = \chi(b) f(g) \ \forall b \in B, g \in G \}$$

Rk: $KZ \backslash G / KZ$ is discrete & infinite

But $B \backslash G$ is cpt
$\cong \mathbb{P}^1(F)$

Thm (Barthel-Livne) $B \backslash G / B$ has 2-elements

If $R = E$, then $\mathrm{Ind}_B^G \chi$ is irreducible

unless $\chi_1 = \chi_2$ in which case $\exists$ 1-dim'l sub.

e.g. if $\chi_1 = \chi_2 = 1$.

then $\mathrm{Ind}_B^G \chi \supset$ cst ftnc as a $G$-inv. sub

& the quotient is irred, call it Steinberg.

[ Rk: No $\mathrm{Ind}_B^G \chi$ has 1-dim quot & a Steinberg sub.

in char 0 case

Haar measure fails
$\mu(\mathbb{Z}_p) = 1$ in char $p$
$\mu(p\mathbb{Z}_p) = \frac{1}{p}$

The irreducible $\mathrm{Ind}_B^G \chi$ are called principal series.

Trick for studying $c\text{-}\mathrm{ind}_{KZ}^G V$

is this: Let $R = \mathcal{H}(V)[\frac{1}{T}] \cong E[T][\frac{1}{T}]$ ← depends on V

B+L write down an $R$-valued char $\underset{\sim}{\chi}$ of $T$.

& prove that for $\dim V > 1$

$\underset{\sim}{\chi} = \underset{\sim}{\chi_1} \underset{\sim}{\chi_2}$

$\underset{\sim}{\chi_1}$ unram. $\pi \to T^{-1}$

$$R \otimes c\text{-}\mathrm{ind}_{KZ}^G V \cong \mathrm{Ind}_B^G(\underset{\sim}{\chi}) \quad \text{as } \underline{R\text{-modules}}!$$

$$\therefore \frac{c\text{-}\mathrm{ind}_{KZ}^G V}{T-\lambda} = \mathrm{Ind}_B^G(\chi \bmod (T-\lambda))$$

<u>Thm</u> (B-L)

If $V = \mathrm{Symm}^r$ & $\lambda \in E$ & $\eta : F^\times \to E^\times$

& $\underline{\lambda \neq 0}$.

then $\pi(\mathrm{Symm}^r, \lambda, \eta)$ is <u>irreducible</u>

unless (i) $\underline{r = 0}$ & $\lambda = \pm 1$ ← 1-dim quot

or (ii) $r = \underline{p-1}$ & $\lambda = \pm 1$ ← 1-dim sub.

in which case $\pi(\mathrm{Symm}^r, \lambda, \eta)$ has a 1-d J-H factor

& a twist of Steinberg as the other.

Hence <u>Rough</u> <u>classification</u>

of mod $p$ smooth irred rep'ns of $GL_2(F)$ with a central character.

1) 1-dim ones
2) principal series
3) twist of Steinberg
4) irred. quotient of $\pi(r, 0, \eta)$

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Happy news for $F = \mathbb{Q}_p$.

Breuil proved $\pi(r, 0, \eta)$ are irred. $\forall r$

$\boxed{n = 2, \ F = \mathbb{Q}_p}$

If $F = \mathbb{Q}_{p^f}$ $\quad f > 1$

then $\pi(r, 0, \eta)$ has got infinite length! $\forall r$

Paskunas tells me that

if $f > 1$.

$\pi(\underline{r}, 0, \eta) \supseteq c\text{-}\mathrm{ind}_{KZ}^{G}(V)$ $\quad V \to \mathrm{Ind}_{B}^{\Gamma}(1 \otimes \omega^{r})$

for some appropriate $V$

$2^f - 2$

Consequence: For $GL_2(\mathbb{Q}_p)$.

We know all irred. rep'ns $/ \overline{\mathbb{F}}_p$.

$\pi(r, \lambda, \eta) \cong \pi(r, -\lambda, \eta \otimes u(-1))$

$u(x) : F^\times \to E^\times$

is unram. char. sending $\pi$ to $x$.

& if $\lambda = 0$.

$\pi(r, 0, \eta) \cong \pi(p-1-r, 0, \eta \cdot \omega^r)$

– How to explame Serre's weight conj via mod p – Langlands

Say $\rho : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{F}}_p)$ irred. & modular, level ($N = N(\rho)$).

& one can define $\Pi(\rho) := \lim H^1(X_1(N; p^r), \overline{\mathbb{F}}_p)[m]$

a rep'n of $GL_2(\mathbb{Q}_p)$

$E$ pts order $N$
2 pts order $p^r$
gen. $E[p^r]$

Idea: $\pi(\rho)$ should depend only on $\rho|D_p$.

(Emerton says that his lectures + Thms of Colmez (unpublished) shd imply this if $End(\rho|D_p) = \overline{\mathbb{F}}_p$ & $\rho|D_p \neq$ twist of $\begin{pmatrix} 1 & * \\ 0 & \omega \end{pmatrix}$)

$\rho$ : modular of level $N$ & wt $\sigma$

$\iff \mathrm{Hom}_\Gamma(\sigma^*, \pi(\rho)^{K(1)}) \neq 0$

& so now one can try & guess what $\Pi(\rho)$ should be, hoping that it's finite length

Serre conj. which $\sigma$ occur

Idea: a given $\rho$. know list of $\sigma$.

$\therefore$ should know $\Pi(\rho)^{K(1)}$

For all reps $\Pi$ in our list, one can compute $\Pi^{K(1)}$ as rep of $\Gamma = GL_2(\mathbb{F}_p)$

& in particular $soc_\Gamma(\Pi^{K(1)}) = \Sigma$ irred subreps of $\Pi^{K(1)}$

1-dim rep'ns - shouldn't show up in $\Pi(\rho)$'s.

P = Principal series, then $P^{K(1)} \supseteq$ exactly one irred rep'n with multi. 1.

except $\mathrm{Ind}_B^G(\chi_1, \chi_2)$ where $\overline{\chi_1} = \overline{\chi_2}$. where you get irred subs. one corresp. to wt 2 & others to wt p+1.

$\chi_i : F^\times \longrightarrow E^\times$

$\overline{\chi_i} : O^\times \longrightarrow E^\times$

$\searrow_{k^\times} \nearrow$

$\Pi(r, 0, \eta)^{K(1)} \supseteq$ two irred rep'ns of $GL_2(\mathbb{F}_p)$ corresponding to wts $k$ & $p+3-k$ $(2 \le k \le p+1)$ (up to twist)

If $\rho|_{I_p} = \begin{pmatrix} \omega_2^{k-1} & 0 \\ 0 & \omega_2^{p(k-1)} \end{pmatrix}$

then Serre predicts wt $k$ & a twist of wt $k'$ $k + k' = p+3$.

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## Breuil's mod p local Langlands Correspondence

If $\rho|_{D_p} = \begin{pmatrix} \omega^{r+1} \cdot u(\lambda^{-1}) & 0 \\ 0 & u(\lambda) \end{pmatrix}$, then define $\Pi_c(\rho)$ conj

$r = k-2$

$\mathrm{Sym}^r$

$\mathrm{Sym}^{k-2}$

$$\Pi_c(\rho) := \left(\Pi(r, \lambda, 1) \oplus \Pi(p-3-r, \lambda^{-1}, \omega^{r+1})\right)^{ss}$$

add p-1 if necessary

$(\Pi_c(\rho))^{K(1)} \supset \sigma \iff$ Serre predicts $\sigma$.

$\rho|_{I_p} = \begin{pmatrix} \omega_2^{r+1} & 0 \\ 0 & \omega_2^{p(r+1)} \end{pmatrix}$ define $\Pi_c(\rho) = \Pi(r, 0, 1)$ $GL_2(\mathbb{Q}_p)$ $\curvearrowright$

Def of $\Pi_c$ is due to Breuil.

Breuil's definition of $\pi_c$ was motivated by an attempt to compute $\varinjlim_r H^1(X_1(N; p^r), \mathbb{Z}_p)[m]$ & its reduction.

Emerton's extension of this idea:

If $\rho|D_p = \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$, $* \neq 0$,

Emerton suggests that $\pi(\rho)$ should be a non-split ext'n of one $\pi$ by the other.

$(\pi(\rho)^{ss})^{K(1)}$ may & will contain more irred subs than $\pi(\rho)^{K(1)}$

If $\rho = \begin{pmatrix} 1 & * \\ & \omega \end{pmatrix}$ then Emerton's $\pi(\rho)$ semi-simplified is strictly bigger than $\pi_c\begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}$.

$\longrightarrow$ If $\rho|D_p$ is not semi-simple, the dust will clear soon.

If $\rho|D_p$ is <u>semi-simple</u>, Emerton can almost prove $\pi(\rho) = \pi_c(\rho)$

<u>Rk</u>: If $\rho|D_p$ is $\chi_1 \oplus \chi_2$ then PS's that Breuil associates to $\rho$ are ones which look isomorphic but aren't.

char 0. $I(\chi_1, \chi_2 |\cdot|) = I(\chi_2, \chi_1 |\cdot|)$

If $\rho|D_p$ is irred. $\pi_c(\rho)$ is irred.

Terrifying fact: for $GL_2(\mathbb{Q}_{p^2})$ things are much more bewildering.

Global side: $M/\mathbb{Q}$
real quad.
$p$ inert in $M$

$\exists$ theory of Hilbert M-Fs give rep's $Gal(\bar{M}/M) \to GL_2(\bar{\mathbb{Q}}_p)$

$\exists$ Serre's conjecture (Diamond) $GL_2(\bar{\mathbb{F}}_p)$

$\exists$ general $\pi(\rho)$ construction using Shimura curve

$\circlearrowleft$
$GL_2(\mathbb{Q}_{p^2})$ $\nexists$ analogue of Colmez' functor

Emerton knows of no strategy for proving $\pi(\rho)$ only ~~pen~~ depends on $\rho|D_p$

Let's attack the problem by listing rep'ns of $GL_2(\mathbb{Q}_{p^2})$
(have a "partial list")

Paskunas' theorem:
write down an irred quotient $PAS(r, o, \eta)$ of $\pi(r, o, \eta)$

$PAS(r, o, \eta)^{K(1)}$ explains 2 wts.

$k_1, k_2$
& a twist of $p+3-k_1$, $p+3-k_2$.

prin. series explains 1 wt.

For $\rho: Gal(\bar{M}/M) \to GL_2(\bar{\mathbb{F}}_p)$ s.t. $\rho|D_p$ is semi-simple
Fred predicts 4 wts
$2^g$

$\rho|D_p$ irred $\longrightarrow$ irred $\pi$ ?

Christophe's last lecture:
a Construction of a $\pi$ s.t. $\pi^{K(1)} \supseteq 4$ irred. subs.

$GL(\mathbb{Q}_{p^2})$ is "done"

$\rho|D_p = \begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix}$ $\quad \pi(\rho) = PS \oplus PS \oplus \underline{PAS}$

$\rho|D_p$ irred : use Breuil / Paskunas new rep'n

<u>Problem</u> : construction seemed to depend on many scalars which don't get fit into picture.

$GL_2(\mathbb{Q}_{p^3})$ : even if $\rho|D_p$ irred, $\pi(\rho)$ will <u>not</u> be.