Apr. 12. 2006. Wed. 1:00pm Matt Emerton. (E.V.)

Local - Global compatibility in p-adic local Langlands 1.

$A = \mathbb{F}_p \quad \mathbb{Z}_p \quad \mathbb{Q}_p \quad \overline{\mathbb{Q}}_p$

$\quad\;\; \mathbb{F}_q \quad \mathcal{O}_E \quad E$

$$H^1_A := \varinjlim_N H^1_{et}(X(N)_{/\overline{\mathbb{Q}}}, A) \circlearrowleft \quad G_{\mathbb{Q}} \times GL_2(\mathbb{A}_f)$$

$W \in \mathrm{Irr}(GL_2)$

$$H^1(V_W)_A := \varinjlim_N H^1_{et}(X(N), V_{W_A}) \qquad A \text{ is } \mathbb{Q}_p, \overline{\mathbb{Q}_p}$$

V_W: local system on $X(N)$

$\circlearrowleft$ $G_{\mathbb{Q}} \times GL_2(\mathbb{A}_f)$

Thm (Deligne, Langlands, Carayol)

$k \geq 2$
$$H^1(V_{W_k})_{\overline{\mathbb{Q}}_p} \cong \bigoplus_{f} \rho_f \otimes \bigotimes_{\ell}' \pi_\ell(f)$$

f: cuspidal new form wt k; $\rho_f \circlearrowleft G_{\mathbb{Q}}$; $\pi_\ell(f) \circlearrowleft GL_2(\mathbb{Q}_\ell)$

$(W_k)_{\overline{\mathbb{Q}}_p} = (\mathrm{Sym}^{k-2} \overline{\mathbb{Q}}_p^2)^\vee$

$$\pi_\ell(f) \xleftrightarrow[\text{via Local Langlands}]{} \rho_f|_{D_\ell}$$

Conj. (F.M. + previous Thm)
(Langlands)

If ρ is an irred. conti. geom. rep'n $\rho: G_{\mathbb{Q}} \longrightarrow GL_2(\overline{\mathbb{Q}}_p)$ w/ distinct H-T wts $(0, 1-k)$, then

$$G_{\mathbb{Q}} \times GL_2(\mathbb{A}_f) \circlearrowright \operatorname{Hom}_{G_{\mathbb{Q}}}(\rho, H^1(V_{w_k})_{\overline{\mathbb{Q}}_p}) \cong \bigotimes_{\ell}{}' \pi_\ell(\rho|D_\ell)$$

$\hat{H}^1_A$:= p-adic completion of H^1_A. $(A = \mathbb{Z}_p, \mathcal{O}_E)$
$\in$ p-adic Banach space

$\hat{H}^1_{\mathbb{Q}_p} := \hat{H}^1_{\mathbb{Z}_p}[\frac{1}{p}]$ (Alternatively: $\hat{H}^1_{\mathbb{Z}_p} = \varprojlim_s H^1_{\mathbb{Z}/p^s\mathbb{Z}}$)
$E \qquad \mathcal{O}_E$
$\circlearrowleft$
$G_{\mathbb{Q}} \times GL_2(\mathbb{A}_f)$

Conj: If ρ is an odd, irreducible, conti rep'n $\rho: G_{\mathbb{Q}} \longrightarrow GL_2(E)$, then

multiplicity space

$$\mathcal{M}(\rho) := \operatorname{Hom}_{G_{\mathbb{Q}}}(\rho, \hat{H}^1_E) \cong \hat{\bigotimes_{\ell}}{}' \hat{\pi}_\ell(\rho|D_\ell)$$

where if $\ell = p$, p-adic Banach space

$\hat{\pi}_p(\rho|D_p)$ is "the" unitary $GL_2(\mathbb{Q}_p)$ – Banach sp. rep'n attached to $\rho|D_p$ via p-adic L.L.

$\ell \neq p$. $\hat{\pi}_\ell(\rho|D_\ell)$ = p-adic completion of $\pi^m_\ell(\rho|D_\ell)$.

where $\pi^m_\ell(\rho|D_\ell) = \pi_\ell(\rho|D_\ell)$ (by L.L)

except if $\pi_\ell(\rho|D_\ell)$ is 1-dim'l, in which case it is the associated (reducible) principal series

twist $\otimes$ $(0 \to St \to \underset{\text{series}}{\text{principal}} \to \mathbb{1} \to 0)$

Consequence of conjecture

(together with standard conjecture about $\hat{\pi}_p$)

- If ρ is odd, irred. geometric with H-T wt $0, 1-k < 0$ then ρ comes from a classical new form of wt k.

- If ρ is abs. irred. unramified at all but fin. many primes and if $\rho|D_p$ is triangulline, then ρ comes from a f.s. o.c. eigen form
  ↳ nontrivial Jacquet module
  p.s.t ⇒ ∃ loc. alg vector

Fix $\bar\rho : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{F}_q)$ conti, abs. irred. "modular" (Serre Conjecture)

Let $X_{\bar\rho,\Sigma} = \mathrm{Spf}\, R_{\bar\rho,\Sigma}$

full universal deformation ring.

$$R_{\bar\rho} := \varprojlim_{\Sigma} R_{\bar\rho,\Sigma} \qquad \mathrm{Spec}\, R_{\bar\rho} = X_{\bar\rho}$$

$$\text{"}\mathrm{Spf}\text{"} R_{\bar\rho} = \varinjlim_{\Sigma} \mathrm{Spf}(R_{\bar\rho,\Sigma})$$

$$\begin{array}{ccc} R_{\bar\rho} & \longrightarrow & \mathbb{T}_{\bar\rho} := \varprojlim_{\Sigma} \mathbb{T}_{\bar\rho,\Sigma} \\ \downarrow & & \downarrow \\ R_{\bar\rho,\Sigma} & \dashrightarrow & \mathbb{T}_{\bar\rho,\Sigma} \end{array}$$

$\bar\rho$: abs. irred

↪ p-adic modular forms conductor inside in Σ.

Form $\hat{H}^1_{\mathcal{O}_E,\bar\rho}$ (summand of $\hat{H}^1$ given by $\bar\rho$) $\quad \mathcal{O}_E \twoheadrightarrow \mathbb{F}_q$

$\circlearrowleft$

$\mathbb{T}_{\bar\rho}[G_{\mathbb{Q}} \times GL_2(\mathbb{A}_f)]$

Thm(?) (Colmez)

Suppose $\bar\rho|D_p \not\cong$ twist $\otimes \begin{pmatrix} 1 & * \\ 0 & \omega \end{pmatrix}$ possibly 0. mod p cyclo

Then. $\exists\ \bar\pi$ s.t $\mathrm{Def}(\bar\pi) \cong \mathrm{Def}(\bar\rho|_{D_p})$

Smooth admissible $GL_2(\mathbb{Q}_p)$ rep'n / $\mathbb{F}_q$

A map of deformation functors.

<u>Thm</u> Given the thm(?) of Colmez and assuming that the work of Berger-Breuil extends to the Frob-<u>non</u> s.s. case.
Then if ρ is pro modular

(i.e. in $\operatorname{Spec} \mathbb{T}_{\bar\rho} \hookrightarrow \operatorname{Spec} R_{\bar\rho}$) and $\rho|D_p$ is irred.
(i.e. ρ <u>not</u> ordinary at p)

then $\hat{\pi}(\rho|D_p) \hookrightarrow \mathcal{M}(\rho)$.

April 19. 2006. Wed. Matt Emerton (2nd lecture)

$$H^1_A = \varinjlim_N H^1_{et}(X_1(N), A)$$

$$\circlearrowleft$$

$$G_{\mathbb{Q}} \times GL_2(\mathbb{A}_f)$$

$\widehat{H}^1_{O_E}$ = p-adic completion of $H^1_{O_E}$

$$\widehat{H}^1_E = E \otimes_{O_E} H^1_{O_E}.$$

<u>Conj</u>: $\rho : G_{\mathbb{Q}} \longrightarrow GL_2(E)$

conti irred. odd.

then $\pi(\rho) := \mathrm{Hom}^{cts}_{G_{\mathbb{Q}}}(\rho, \widehat{H}^1_E)$ $\circlearrowleft GL_2(\mathbb{A}_f)$

$$\overset{?}{\cong} \widehat{\bigotimes_{\ell}} \widehat{\pi}_\ell(\rho|D_\ell)$$

<u>Thm (?) of Colmez</u>

If $\bar{\rho} \not\cong \text{twist} \otimes \begin{pmatrix} 1 & * \\ 0 & \omega \end{pmatrix}$, then $\exists\ \bar{\pi} \circlearrowleft GL_2(\mathbb{Q}_p)$

local

mod p local Langlands

s.t. $\mathrm{Def}(\bar{\rho}) \cong \mathrm{Def}(\bar{\pi})$

$$\rho \longleftrightarrow \widehat{\pi}(\rho)$$

with the property$^{\text{ies}}$ that (a) if ρ is a Δ'line lift to E,

then $\hat{\pi}(\rho)$ is the E-Banach $GL_2(\mathbb{Q}_p)$-rep'n attached to ρ via the Δ'line p-adic L-L.

(b) If ρ is p.s.t. different H-T weights, then $\hat{\pi}(\rho) \neq 0$ loc.alg

<u>Thm 1</u> (Last time)

Given <u>Thm (?)</u> of Colmez, if ρ is global. promodular of tame level K^p. $\bar{\rho}$ is abs. irred.

$\bar{\rho}|D_p \neq \text{twist} \otimes \begin{pmatrix} 1 & * \\ 0 & \omega \end{pmatrix}$ $\rho|D_p$ is non-split, then
indecomposable

$$\exists\ \hat{\pi}(\rho|D_p) \underset{GL_2(\mathbb{Q}_p)\text{-equiv.}}{\hookrightarrow} \mathcal{U}(\rho)^{K^p}$$

<u>Thm</u> (Böckle)

If $p > 2$. $\bar{\rho} : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{F}_p)$ cont. irred. modular s.t.

$\bar{\rho}|_{G_{\mathbb{Q}(\sqrt{\pm p})}}$ is irred. $\bar{\rho}|D_p \neq \text{twist} \otimes \begin{pmatrix} 1 & * \\ 0 & \omega \end{pmatrix}$.

(a) $\bar{\rho}|D_p$ is reducible and p-distinguished.

(b) $\bar{\rho}|D_p$ is flat up to twist

then any lift of $\bar{\rho}$ unram. at all but fin. many primes is promodular.

<u>Stronger Conjecture.</u>

Fix. <u>global $\boxed{\bar{\rho}}$</u> (odd. irred. contn modular)

<u>Last time</u>: $\hat{H}^1_{\bar{\rho}}$

$\circlearrowleft$

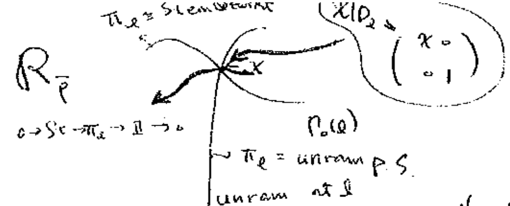
$R_{\bar{\rho}} \twoheadrightarrow \mathbb{T}_{\bar{\rho}}$
$G_{\mathbb{Q}}$

$R_{\bar{\rho}}$ $\quad \mathbb{T}_{\bar{\rho}}[G_{\mathbb{Q}} \times GL_2(\mathbb{A}_f)]$

<u>Conj</u>: Assume $\bar{\rho}|D_p \neq \text{twist} \otimes \begin{pmatrix} 1 & * \\ 0 & \omega \end{pmatrix}$

Then $\hat{H}^1_{\bar{\rho}} \cong \rho^{univ} \otimes_{R_{\bar{\rho}}} \hat{\pi}(\rho^{univ}|_{D_p}) \otimes X$

an explicit $R_{\bar{\rho}}[GL_2(\mathbb{A}_f^p)]$-module, which interpolates ℓ-adic L.L over Spec $R_{\bar{\rho}}$ for all $\ell \neq p$



$\boxed{\text{Thm}}$ Assume that Thm (?) of Colmez, Berger-Breuil extends to compatible Frob-non-semi-simple case

and $\bar\rho|D_p \not\simeq \text{twist} \otimes \begin{pmatrix}1 & *\\ 0 & \omega\end{pmatrix}$

and $\mathrm{End}_{D_p}(\bar\rho|D_p) = \mathbb{F}_q$. then Stronger Conjecture is true.

Consequences of Stronger Conj.

- p-adic L-ftns over deformation space / eigen curve.
- Mod p local-global compatibility:

$$\mathcal{M}(\bar\rho) := \mathrm{Hom}_{G_{\mathbb{Q}}}(\bar\rho, H^1_{\mathbb{F}_q}) \simeq \bar\pi \otimes \bigotimes_{\ell\neq p}' \bar\pi_\ell(\bar\rho|D_\ell)$$

Sketch of proof

$$\hat H^1_{\bar\rho} \simeq \rho^{univ} \otimes \hat\pi(\rho^{univ}|D_p) \hat\otimes X$$

$$H^1_{\mathbb{F}_q}[m] \simeq \bar\rho \otimes \bar\pi \otimes X[m]$$

$$\mathcal{M}(\bar\rho) = \bar\pi \otimes X[m]$$

$$\mathcal{M}(\bar\rho)^{I(1)} = \bar\pi^{I(1)} \otimes X[m]$$

Sketch of pf of two theorems

Assume $\bar\rho$: global abs irred modular

$$\bar\rho|D_p \not\simeq \begin{pmatrix}1 & *\\ 0 & \omega\end{pmatrix} \otimes \text{twist}.$$

Define

$$X := \mathrm{Hom}_{R_{\bar\rho}[G_{\mathbb{Q}} \times GL_2(\mathbb{Q}_p)]}\left(\rho^{univ} \otimes \hat\pi(\rho^{univ}|D_p), \hat H^1_{\bar\rho}\right)$$

Then $\rho^{univ} \otimes \hat\pi(\rho^{univ}|D_p) \hat\otimes X \xrightarrow{\text{eval.}} \hat H^1_{\bar\rho}$

$\rho : G_{\mathbb{Q}} \to GL_2(E)$ lifting $\bar{\rho}$

Let $\mathfrak{P}_\rho \in \operatorname{Spec} R_{\bar{\rho}}$ be the kernel of $R_{\bar{\rho}} \to E$ classifying ρ

$$X[\mathfrak{P}_\rho] \overset{\text{by def}^{\text{n}}}{=} \operatorname{Hom}(\rho \otimes \hat{\pi}(\rho)_{\mathfrak{p}_p}, \hat{H}'_{\bar{\rho}}[\mathfrak{P}_\rho])$$

$$X[\mathfrak{P}_\rho] \neq 0 \iff \exists\ \hat{\pi}(\rho|_{D_p}) \xrightarrow[\neq 0]{} \mathcal{U}(\rho)$$

$$X[\mathfrak{P}_\rho]' = ((X')/\mathfrak{P}_\rho) / \text{torsion}$$

$$(X')^{K^p} = (X^{K^p})'$$

$$X^{K^p}/p\,[\mathfrak{m}] \hookrightarrow \operatorname{Hom}(\bar{\rho} \otimes \bar{\pi}, (H^1_{\mathbb{F}_q})^{K^p})$$

fin. dim. by admissiblity

April 21. 2006. Friday. 1:00 PM ~ 2:00 PM

Matt Emerton. (3rd lecture)

Last time: $\bar{\rho}: G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{F}_q)$ $\quad$ $\begin{array}{ccc} O_E & \hookrightarrow & E \\ \downarrow & & \\ \mathbb{F}_q & & \end{array}$

irred, modular

$R_{\bar{\rho}} \twoheadrightarrow \mathbb{T}_{\bar{\rho}}$

$\mathrm{Spec}\, \mathbb{T}_{\bar{\rho}} \hookrightarrow \mathrm{Spec}\, R_{\bar{\rho}}$ $\qquad$ $\bar{\rho}|_{D_p} \neq \text{twist} \otimes \begin{pmatrix} 1 & * \\ 0 & \omega \end{pmatrix}$

the full deformation ring parametrizing all deformation.

pro modular deformation (from Katz p-adic modular form)

$$\begin{array}{c|c} \mathbb{C}\ \tilde{\hat{H}}^1_{\bar\rho} & \rho^{univ} \sim \rho^{univ}|D_p \overset{\text{Colmez}^{(?)}}{\longleftrightarrow} \hat\pi(\rho^{univ}|D_p) \\ \mathbb{T}_p[G_{\mathbb{Q}} \times GL_2(\mathbb{A}_f)] & R_{\bar\rho} \qquad\qquad R_{\bar\rho} \end{array}$$

- $X := \mathrm{Hom}_{R_{\bar\rho}[G_{\mathbb{Q}} \times GL_2(\mathbb{Q}_p)]}\left(\rho^{univ} \otimes \hat\pi(\rho|D_p), \hat{H}^1\right)$ $\circlearrowleft GL_2(\mathbb{A}_f^p)$
- X^{K^p} is cofinitely generated over $R_{\bar\rho}$ $\forall K^p \underset{\text{cpt open}}{\subseteq} GL_2(\mathbb{A}_f^p)$
- X^{K^p} is supported on $\mathrm{Spec}(\mathbb{T}_{\bar\rho}(K^p))$ $\rightarrow$ $\boxed{\hat\pi(\rho) \xrightarrow{\neq 0} \mathcal{U}(\rho) \ \forall \text{ promodular } \rho}$
- $\rho^{univ} \otimes \hat\pi(\rho^{univ}|_{D_p}) \hat\otimes X \xrightarrow{\text{eval}} \hat{H}^1_{\bar\rho}$ is onto after $\otimes_{\mathcal{O}_E} E$
- If $\mathrm{End}_{D_p}(\bar\rho|D_p)$ $\left(= \mathrm{End}_{GL_2(\mathbb{Q}_p)}((\bar\pi))\right) = \mathbb{F}_q$ $\quad \mathbb{P}^1_{/\mathbb{F}_q}$

  Then the map "eval" is injective after $\otimes_{\mathcal{O}_E} \mathbb{F}_q$ $\Longrightarrow$ "eval" is an $\cong$.

$\boxed{\mathbb{1} \cdot St \cdot \mathrm{Ind}_B^G\, \omega\otimes\omega^{-1}}$

f : cuspidal new form of wt $k \geq 2$,
tame conductor N, defined over E,
with associated $G_{\mathbb{Q}}$-rep'n ρ.

$$\left(\hat{H}^1_{c,E}\right)^{\Gamma_1(N)} \xrightarrow{\{0,\infty\}} E$$

$$W_k = (\mathrm{Sym}^{k-2} E)^\vee$$

$\rho \otimes \pi_p(f) \otimes W_k$ $\rightarrow$ $G_{\mathbb{Q}} \times GL_2(\mathbb{Q}_p)$-equivariant

unique up to scaling.

$\pm$ eigen spaces
for complex conj
are each one dim'l

$$\underbrace{\pi_p(f) \otimes W_k}_{\text{local}} \xrightarrow{\mathcal{U}_f^\pm \ \rightarrow \text{global}} E$$

$\cup\!\!|$ Hodge-Tate Kirillov model

locally const $\rightarrow$ $\mathcal{C}_c^{fin}(\mathbb{Q}_p^\times) \otimes W_k = \mathcal{C}_c^{\leq k-2}(\mathbb{Q}_p^\times) \supseteq \mathcal{C}_c^{\leq k-2}(\mathbb{Z}_p^\times)$ $\leftarrow$ locally poly.

$\boxed{\text{Prop}}$. If $\chi : \mathbb{Z}_p^\times \to E^\times$ is <u>finite</u> <u>order</u>, $0 \le i \le k-2$

$$\int_{\mathbb{Z}_p^\times} \chi(z) \cdot z^i \, d\mu_f^\pm = \underset{\text{normalized}}{\text{alg part}} \text{ of } L(f \otimes \chi, i+1)$$

Granting the local-global compatibility conjecture we see that $\mu_f^\pm$ extends to a continuous fctnal

$$\hat{\pi}(\rho|D_p) \xrightarrow{\mu_f^\pm} E$$

If f is <u>finite slope</u>
Say non-ordinary at p. ($\Rightarrow \rho|D_p$ irred.)
and if g is a p-stabilization of f with $U_p g = \alpha g$

$0 < v_p(\alpha) < k-1$.

$$\Pi_p(f) \otimes W_k$$
$$\|$$
$$\left(\mathrm{Ind}_{\overline{B}}^{GL_2} ur(\alpha) \otimes \psi\right)_{sm} \otimes W_k$$
$$\|$$
$$\left(\mathrm{Ind}_{\overline{B}}^{GL_2} ur(\alpha) \otimes \psi \cdot z^{2-k}\right)_{\text{localg}}$$

$$\hat{\pi}(\rho|D_p) = \overbrace{\left(\mathrm{Ind}_{\overline{B}}^{G} ur(\alpha) \otimes \psi z^{2-k}\right)_{\text{localg}}}^{\text{universal unitary completion}}$$
$$\uparrow$$
$$\left(\mathrm{Ind}_{\overline{B}}^{G} ur(\alpha) \otimes \psi z^{2-k}\right)_{an}$$
$$\cup\!|$$
$$\mathcal{C}^{an}_{\pm}(\mathbb{Z}_p^\times, E)$$

$\boxed{\text{Prop}}$. The composite

$$\mathcal{C}^{an}(\mathbb{Z}_p^\times, E) \subseteq \widehat{\mathrm{Ind}} = \hat{\pi}(\rho|D_p) \xrightarrow{\mu_f^\pm} E$$

is the "α-branch" of the p-adic L-function

$\boxed{\text{Prop}}$. If $\rho|D_p$ is semi-stable, "the fact that $\mu_f^\pm$ factors through $\hat{\pi}(\rho|D_p)$" $\Rightarrow$ M-T-T conjecture

$$\hat{\pi}(\rho|D_p)^{*\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}}$$
$$\|\wr$$
$$D(\rho|D_p)^{\psi=1}$$

local H^1 - Iwasawa

$$(\hat{H}^1_{\bar\rho}) \cong \rho^{univ} \otimes \hat{\pi}(\rho^{univ}|_{D_p}) \tilde{\otimes} X$$

Fix an integral Λ quotient $\mathbb{T}_{\bar\rho} \twoheadrightarrow A$ (kernel $= \mathfrak{p}$) of tame level $\Gamma_1(N)$

$$\begin{tikzcd}
\hat{H}_{\bar\rho}[\mathfrak{p}]^{\Gamma_1(N)} \arrow[dr, "\{0,\infty\}"'] & \cong & \rho^{univ}|_A \otimes \hat{\pi}(\rho^{univ}|_{D_p})|_A \otimes A \arrow[dl, "\mu"] \\
& \mathcal{O}_E &
\end{tikzcd}$$

$\mu^\pm \in A$-dual of $\hat{\pi}(\rho^{univ}|_{D_p})|_A$

mum way.

April 25. 2006. Tuesday. Matt Emerton. (4th lecture)

Mod p Langlands at primes $\ell \neq p$

Goals: State a mod p Langlands conj for 2-dim'l rep'ns of $G_{\mathbb{Q}_\ell}$ $(\ell \neq p)$.

Prove local-global compatibility for mod p modular forms

1) Mod p L.L. Conj

Let $\bar{\rho} : G_{\mathbb{Q}_\ell} \xrightarrow[\text{contr}]{} GL_2(\overline{\mathbb{F}}_p)$

We say $\bar{\rho}$ is minimal (primitive) if $\mathrm{cond}(\bar{\rho}) \leq \mathrm{cond}(\bar{\rho} \otimes \chi)$

$\forall\, \chi : G_{\mathbb{Q}_\ell} \to \overline{\mathbb{F}}_p^{\times}$.

$\rho : G_{\mathbb{Q}_\ell} \longrightarrow GL_2(\overline{\mathbb{Q}}_p)$ "Lifts $\overline{\rho}$" if it admits a $\overline{\mathbb{Z}}_p$-lattice with red'n $= \overline{\rho}$.

<u>Thm</u>: $\exists$ a uniquely determined map $\overline{\rho} \longrightarrow \overline{\pi}(\overline{\rho})$

$$\{\cong \text{ classes of } \overline{\rho}\} \longrightarrow \left\{ \begin{array}{l} \cong \text{ classes of smooth} \\ GL_2(\mathbb{Q}_\ell) - \text{rep'ns} / \overline{\mathbb{F}}_p \end{array} \right\}$$

s.t.
1) $\overline{\pi}(\overline{\rho})$ contains no 1-dim subrep
2) $\forall \chi : G_{\mathbb{Q}_\ell} \longrightarrow \overline{\mathbb{F}}_p^\times$, $\overline{\pi}(\overline{\rho} \otimes \chi) = \overline{\pi}(\overline{\rho}) \otimes (\chi \circ \det)$
3) If $\overline{\rho}$ is minimal and ρ Lifts $\overline{\rho}$ with $\mathrm{cond}(\rho) = \mathrm{cond}(\overline{\rho})$,

then $\pi(\rho)$ admits a lattice with reduction equal to $\overline{\pi}(\overline{\rho})$ (The lattice is then unique)

Here $\rho \longmapsto \pi(\rho)$ is the usual Local Langlands normalized via $\begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix}$ w/ $\chi_1 \cdot \chi_2^{-1} \neq |\cdot|^{\pm 1} = \mathrm{cyclo}^{\pm 1}$.

Then $\pi(\rho) = \mathrm{Ind}_{\overline{B}}^{GL_2} \chi_1 \otimes \chi_2 |\cdot| \cong \mathrm{Ind}_B^G \chi_1 |\cdot| \otimes \chi_2$

2) <u>Local - global compatibility.</u>

Let $\overline{\rho} : G_{\mathbb{Q}} \longrightarrow GL_2(\overline{\mathbb{F}}_p)$
irred, modular

Let $k =$ Serre wt of $\overline{\rho}$, assume $2 \le k \le p+1$.

If $\overline{\rho}|_{D_p}$ is reducible, fix an eigenvalue α of Frob^{-1} on $\overline{\rho}^{I_p}$

Otherwise set $\alpha = 0$.

Consider $\overline{M}_k \leftarrow$ mod forms of wt $k / \overline{\mathbb{F}}_p$.

action of $GL_2(\mathbb{A}_f^p)$

$$\varinjlim_{(p,N)=1} H^0(X(N)/_{\overline{\mathbb{F}}_p}, \omega^{\otimes k}) = \overline{M}_k(N)$$

At level N, $\mathbb{Z}_p[[T_\ell]]_{\ell \nmid Np} \longrightarrow \mathrm{End}(\overline{\mathcal{M}}_k(N))$ with image $\mathbb{T}(N)$

Let $\mathfrak{m}_N = (T_\ell - \mathrm{trace}|_{\bar\rho}\, \mathrm{Frob}_\ell^{-1})_{\ell \nmid Np} \subseteq \mathbb{T}(N)$.

either maximal or the unit ideal

$$\overline{\mathcal{M}}_k \supseteq \overline{\mathcal{M}}_k[\mathfrak{m}] = \varinjlim_N \overline{\mathcal{M}}_k[\mathfrak{m}_N]$$

Thm.

$$\overline{\mathcal{M}}_k[\mathfrak{m}]^{U_p=\alpha} \cong \bigotimes_{\ell \neq p}{}' \, \overline{\pi}(\bar\rho|_{G_{\mathbb{Q}_\ell}})$$

$GL_2(\mathbb{A}_f^p)$ acts on the left; $GL_2(\mathbb{A}_f^p)$-equivariant.

3) Proof of 1). Preliminaries on smooth rep'ns.

Let F be an alg. closed field of char 0 or $p \neq \ell$.

$$G = GL_2(\mathbb{Q}_\ell) \supseteq P = \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \supseteq N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \cong \mathbb{Q}_\ell$$

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \leftarrow x.$$

$$N \cong \mathbb{Q}_\ell = \bigcup_{n \geq 0} \underbrace{\ell^{-n}\mathbb{Z}_\ell}_{\text{pro-}\ell \text{ groups}}$$

Fix $\psi_0 : \mathbb{Q}_\ell \longrightarrow F^\times$ with kernel equal to $\mathbb{Z}_\ell$

Then any non-trivial char's (i.e. $\mathbb{Q}_\ell/\mathbb{Z}_\ell \xrightarrow[\sim]{\psi_0} \mu_{\ell^\infty}(F^\times)$)

$\psi : \mathbb{Q}_\ell \to F^\times$ has the form $\psi(x) = \psi_0(ax)$ $\exists!\, a \in \mathbb{Q}_\ell^\times$

Let V be a smooth P-rep'n $/F$.

If $\psi : \mathbb{Q}_\ell \to F^\times$, V_ψ = max'l quotient of V on which N acts via ψ.

Jacquet module $= V/V(\psi)$

$$V(\psi) = \left\{ v \in V \;\middle|\; \int_{N_0} \psi^{-1}(n)\, n \cdot v \, dn = 0 \text{ for some } \underset{\substack{\text{compact}\\ \text{open}}}{N_0 \subseteq N} \right\}$$

$$= \langle nv - \psi(n) \cdot v \mid v \in V \rangle$$

If ψ = trivial char. $V_\psi = V_N$ ← Jacquet module

$$V(\psi) =: V(N)$$

If ψ is non-trivial. so $\psi(x) = \psi_0(ax)$. $\exists a \in \mathbb{Q}_\ell^\times$.

then $V \xrightarrow[\sim]{\begin{pmatrix} a & \\ & 1 \end{pmatrix}} V$ induces $V_{\psi_0} \xrightarrow{\sim} V_\psi$

$$\left(\text{b/c} \quad \begin{pmatrix} a & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & ax \\ 0 & 1 \end{pmatrix} \right)$$

$$v \longmapsto v_{\psi_0}$$

$$P \circlearrowright V \longrightarrow V_\psi$$

N-equivariant map ↑ N acts via ψ

$$P = \begin{pmatrix} \mathbb{Q}_\ell^\times & \\ & 1 \end{pmatrix} \ltimes N$$

∴ Frobenius Reciprocity

$$V \xrightarrow[P\text{-equiv.}]{} \mathrm{Ind}_N^P V_{\psi_0} \qquad v \longmapsto \left(\begin{matrix} P \\ \rotatebox{90}{$\in$} \\ g \end{matrix} \longmapsto (gv)_{\psi_0} \right)$$

Smooth induction

b/c $P = \begin{pmatrix} \mathbb{Q}_\ell^\times & \\ & 1 \end{pmatrix} \cdot N$

$$\mathrm{Ind}_N^P V_{\psi_0} \cong \mathcal{C}^{sm}(\mathbb{Q}_\ell^\times, V_{\psi_0})$$

$$\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix} f \right)(x) = f(ax)$$

$$\left(\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} f \right)(x) = \psi_0(zx) \cdot f(x)$$

$\boxed{\text{Lemma}}$

i) image of $V \longrightarrow \mathcal{C}^{sm}(\mathbb{Q}_\ell^\times, V_{\psi_0})$ is contained in ftns that vanishes at ∞.

ii) kernel $= V^N$

Pf). i) If $f \in \mathcal{C}^{sm}(\mathbb{Q}_\ell^\times, V_{\psi_0})$ that is N-smooth then for $a \in \mathbb{Q}_\ell^\times$ with $|a| >> 0$

$\begin{pmatrix} a & \\ & 1 \end{pmatrix} f$ is killed by $\begin{pmatrix} 1 & \frac{1}{\ell}\mathbb{Z}_\ell \\ 0 & 1 \end{pmatrix}$, and $\displaystyle\int_{\begin{pmatrix} 1 & \frac{1}{\ell}\mathbb{Z}_\ell \\ 0 & 1 \end{pmatrix}} \psi_0(n)^{-1} n f \, dn = 0$

$\therefore f(a) \equiv 0 \quad \left(f \in \left(\mathrm{Ind}_N^P V_{\psi_0}\right)(\psi_0) = 0 \right)$

ii) <u>Fourier</u> <u>Theory</u>.

$$\left(\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} r \right)_{\psi_0} = 0 \right)$$
$$\overset{\shortparallel}{v}_{\psi_0(a-)}$$

$\boxed{\text{Lemma}}$ ∘ $V(N)_{\psi_0} \xrightarrow{\sim} V_{\psi_0}$

$V(N)$
$:= V(\psi_0)_{\text{trivial}}$
$= \ker(V \to V_N)$

∘ $V(N) \cap V^N = 0$

∘ $V(N) \underset{P\text{-equiv}}{\xrightarrow{\sim}} \mathcal{C}_c^{sm}(\mathbb{Q}_\ell^\times, V(N)_{\psi_0}) \xrightarrow{\sim} \mathcal{C}_c^{sm}(\mathbb{Q}_\ell^\times, V_{\psi_0})$

<u>pf</u>) ∘ $V(N) \subseteq V$. $\therefore V(N)_{\psi_0} \subseteq V_{\psi_0}$

Given $v \in V$, choose n s.t. $\psi_0(n) \neq 1$, then

$$\underset{\substack{\uparrow \\ \text{element of } V(N)}}{(nv - v)_{\psi_0}} = \underset{\substack{\shortparallel \\ 0 \\ \neq}}{(\psi_0(n) - 1)} v_{\psi_0} \quad \therefore v_{\psi_0} \text{ is in image of}$$

$V(N)_{\psi_0} \to V_{\psi_0}$

∘ Let $v \in V(N) \cap V^N$, then for some N_0

$$v \overset{v \in V^N}{=} \int_{N_0} nv \, dn \overset{v \in V(N)}{=} 0$$

∘ If $f \in \mathcal{C}^{sm}(\mathbb{Q}_\ell^\times, V_{\psi_0})$, then $\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} f - f \right)(z)$

$$\shortparallel$$
$$(\psi_0(zx) - 1) f(z)$$

which vanishes if z is close to zero.

$\therefore V(N) \longrightarrow \mathcal{C}^{sm}(\mathbb{Q}_\ell^\times, V_{\psi_0})$ image is in $\mathcal{C}_c^{sm}(\mathbb{Q}_\ell^\times, V_{\psi_0})$

$$\begin{array}{ccc} V(N) & \longrightarrow & C^{sm}(\mathbb{Q}_\ell^\times, V(N)_{\psi_0}) \\ \cap\!\! \mid & \searrow C_c^{sm}(\mathbb{Q}_\ell^\times, V_{\psi_0}) & \downarrow \wr \\ V & \longrightarrow & \subseteq\ C^{sm}(\mathbb{Q}_\ell^\times, V_{\psi_0}) \end{array}$$

$$\therefore\ V(N) \subseteq C_c^{sm}(\mathbb{Q}_\ell^\times, V(N)_{\psi_0}) = C_c^{sm}(\mathbb{Q}_\ell^\times, V_{\psi_0})$$

$$\begin{array}{c} \text{SII} \\ \longrightarrow C_c^{sm}(\mathbb{Q}_\ell^\times . \psi_0) \otimes V(N)_{\psi_0} \end{array}$$

Tmed as P-rep'n
(check!)

P-rep'n V

P acts via action of $\mathbb{Q}_\ell^\times$

$$0 \longrightarrow V(N) \longrightarrow V \longrightarrow V_N \longrightarrow 0$$

$$\downarrow S$$

$$C_c^{sm}(\mathbb{Q}_\ell^\times . \psi_0) \otimes V(N)_{\psi}$$

$V^N \subseteq V_N$?

Lemma. If V is a smooth G-rep'n, then $V^N = V^{SL_2(\mathbb{Q}_\ell)}$

proof) If H is any cpt open s.g. of G,

$(\text{gp gen. by } H \text{ \& } N) \supseteq SL_2$

Thm. If V is an irreducible smooth G-rep'n of infinite dim, ($\Leftrightarrow$ not of the form $\chi \circ \det$) then V_{ψ_0} is 1-dim'l (Uniqueness of Whittaker models)

If $V_{\psi_0} \neq 0$. we say V is generic ($\Leftrightarrow$ inf dim)

$$\Updownarrow$$

$$V \underset{P\text{-equiv.}}{\hookrightarrow} C^{sm}(\mathbb{Q}_\ell^\times, \psi_0)$$

$$\left[\begin{array}{c} \mathrm{Ind}_N^P \psi_0 \\ \mathrm{Ind}_N^G \psi_0 \end{array}\right]$$

$$0 \to V(N) \hookrightarrow V \to V_N \to 0$$

$$V(N) \cong C_c^{\mathrm{sm}}(\mathbb{Q}_\ell^\times, \psi_0) \subseteq C^{\mathrm{sm}}(\mathbb{Q}_\ell^\times, \psi_0) \leftrightarrow V$$

$\boxed{\text{Prop}}$: V_N is always

0 (V cuspidal) → all the others (intrinsically, the matrix coeff. ~~are~~ compactly supp. mod center.)

1 (special or $\chi \circ \det$) dim'l.

2 (principal series)

$St_N = |\cdot| \otimes |\cdot|^{-1}$ $\qquad (\chi \circ \det)_N = \chi \circ \det$

$\mathrm{Ind}_{\bar{B}}^{G}(\chi_1 \otimes \chi_2) \sim (\mathrm{Ind})_N^{ss} = \chi_1|\cdot| \otimes \chi_2|\cdot|^{-1} \oplus \chi_2 \otimes \chi_1$

irred if $\chi_1 \cdot \chi_2^{-1} \neq 1, |\cdot|^{2}$

twist of

$St := (\mathrm{Ind}_{\bar{B}}^{GL_2} \mathbb{1} \otimes \mathbb{1})_{\mathbb{1}}$

irred if $\ell \not\equiv -1 \, (p)$

$(\ell \equiv -1 \, (p))$ $\quad 0 \to \text{cuspidal} \to St \to |\cdot| \cdot |\circ \det \, (\to 0)$

quadratic char

April 26. 2006. Wed. 1pm. Matt Emerton
(5th lecture)

Yesterday 2 Things : 1) Local correspondence
2) Local - Global compatibility.

Construction of local correspondence

Lattices in $GL_2(\mathbb{Q}_\ell)$ - rep'ns / $\overline{\mathbb{Q}}_p$ $(\ell \neq p)$ — Vigneras

Given $V/\overline{\mathbb{Q}}_p$, When does $\exists$ a p-adically separated $GL_2(\mathbb{Q}_\ell)$ - invariant lattice in V ? — Compositio '89

If V is p.s $\mathrm{Ind}_{\overline{B}}^{G} \chi_1 \otimes \chi_2$, we need χ_1, χ_2 unitary

If special, cuspidal, need unitary central character

$$V \cong C_c^{sm}(\mathbb{Q}_\ell^\times, \overline{\mathbb{Q}}_p)$$

Suppose V admits a lattice V_0. ψ_0

Consider $\overline{V_0} = V_0/\mathfrak{m}V_0$

maximal ideal in $\overline{\mathbb{Z}}_p$

1) If V is cuspidal, $\overline{V_0}$ is cuspidal irred.

2) If V is special, either $\overline{V_0}$ is special irred ($\ell \not\equiv -1$ (p))

or $\ell \equiv -1$ (p), J-H factors are cuspidal, $\mathbb{1}$-dim

3) If V is p.s.

$\overline{V_0}$ is $\begin{cases} \text{p.s. irred} \\ \ell \equiv 1 \text{ (p) with } p \neq \ell \text{ \& } \overline{V_0} \text{ has J.H factors special, 1-dim'l.} \\ \ell \equiv -1 (p), \text{ can have that } \overline{V_0} \text{ has 3 J-H. factors 1 cuspidal, 2 1-dim.} \end{cases}$

In particular, $\overline{V_0}$ has a unique generic J-H factor.

Ribet's lemma $\Rightarrow$ Choose Lattice V_1

s.t. $\overline{V_1}$ has generic as a sub, no 1-dim subs

$\overline{V_1} \to \overline{V_2}$

V_1 is unique up to scaling

$e_1, \dots, e_f \; e_{f+1}, \dots, e_d$

$\pi V_0 \subseteq M \subseteq V_0$

$0 \to \overline{\text{gen}} \to \overline{V_0} \to \overline{Q} \to 0$

$\overline{Q} \supseteq \mathbb{1}$

$0 \to \overline{\text{gen}} \oplus \mathbb{1} \subseteq \overline{V_0} \to \chi \to 0$

$$\begin{pmatrix} x & 0 & \boxed{x} \\ 0 & x & x \\ 0 & 0 & x \end{pmatrix} \to \begin{array}{c} \qquad\qquad \text{unchanged} \\ \begin{pmatrix} x & x & \boxed{x} \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix} \end{array}$$

$\overline{\rho} : G_{\mathbb{Q}_\ell} \longrightarrow GL_2(\overline{\mathbb{F}}_p)$

Twist $\overline{\rho}$ to be minimal conductor

Lift $\overline{\rho}$ to ρ with $\text{cond}(\overline{\rho}) = \text{cond}(\rho)$

Take $V = \pi(\rho)$, set $\overline{\pi}(\overline{\rho}) = \overline{V_1}$

- $\bar\rho$ is irreducible, $\bar\pi(\bar\rho)$ cuspidal
- $\bar\rho$ reducible, up to a twist

$$\bar\rho^{ss} = \mathbb{1} \oplus \psi, \quad \mathbb{1} \subseteq \bar\rho$$

$\mathrm{Ext}^1_{G_{\mathbb{Q}_\ell}}(\psi, \mathbb{1}) = H^1(G_{\mathbb{Q}_\ell}, \psi^{-1})$ has dimension $\begin{cases} 0 & \text{if } \psi \neq \mathbb{1}, \text{cycl}^{-1} \\ 1 & \text{if } \psi = \mathbb{1}, \text{cyclo}^{-1} (\ell \not\equiv 1 \ (p)) \\ 2 & \text{if } \psi = \mathbb{1} = \text{cyclo}^{-1} (\ell \equiv 1 \ (p)) \end{cases}$

If $\bar\rho = \mathbb{1} \oplus \psi$, take $\rho = \mathbb{1} \oplus \tilde\psi$ (Teich lift.)

$$\bar\pi = \begin{cases} \mathrm{Ind}_B^G \, \mathbb{1} \otimes \psi|\,| & \text{if } \psi \neq \text{cyclo}^{-1} \\ 0 \to St \to \bar\pi \to \mathbb{1} \to 0 & \text{if } \psi = \text{cyclo}^{-1},\ \ell \not\equiv -1 \ (p) \\ 0 \to \text{cuspidal} \to \bar\pi \to 2\text{-dim} \to 0 & \text{if } \psi = \text{cyclo}^{-1},\ \ell \equiv -1 \ (p) \end{cases}$$

If $\bar\rho$ is an unramified ext'n of $\mathbb{1}$ by $\mathbb{1}$.

$$\bar\pi(\bar\rho) = \bar\pi(\mathbb{1} \oplus \mathbb{1})$$

If $\bar\rho$ is an ext'n of cyclo^{-1} by $\mathbb{1}$ that is (tamely) ramified,

$$\bar\pi(\bar\rho) = \begin{cases} St & \text{if } \ell \not\equiv -1 \ (p) \\ 0 \to \text{cuspid} \to \bar\pi \to |\,|\circ\det \to 0 & \text{if } \ell \equiv -1 \ (p) \end{cases}$$

Proof of local-Global compatibility

$$\bar{\mathcal{U}}_k[m]^{c_p = \alpha} \overset{2 \le k \le p+1}{=} \bigotimes_{\ell \neq p}{}' \bar\pi(\bar\rho|_{D_\ell})$$

Existence of minimal lift (weak Serre $\Rightarrow$ Strong Serre)

known LHS $\supseteq$ RHS

Theory of new-forms $\Rightarrow$ generic JH factors coincide

Kirillov model $\Rightarrow$ (LHS = RHS)

Theory of Jacquet modules $\Rightarrow$ $\mathbb{1}$-dim JH factors coincide

$\rho_m : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{T}(N)_m)$. $q \| N$

lifting $\bar\rho$ $\quad U_q$

$$M = \overline{M}_k(\ell^\infty N')[m]^{U_p = \alpha}$$

$\circlearrowleft$ $\quad$ N': ℓ-dependent Artin conductor of $\bar\rho$.

$GL_2(\mathbb{Q}_\ell)$

$$M \longrightarrow C^{sm}(\mathbb{Q}_\ell^\times, M_{\psi_0})$$

$$\| $$

$$C^{sm}(\mathbb{Q}_\ell^\times) \otimes M_{\psi_0}$$

$$\omega$$

$$\mathbf{1}_{\mathbb{Z}_\ell^\times}$$

$$0 \longrightarrow \bar\pi(\bar\rho|D_\ell) \subseteq M \twoheadrightarrow \underset{\cancel{0}}{\overset{\chi_0 \det \atop \cap}{Q}} \longrightarrow 0$$

finite dimensional