Tilouine's lecture 2/6

Siegel Modular Forms & Galois repr.

1) Modularity for GL_2

$N \geq 1$, $\Gamma_1(N) = \{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z});\ \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod N \}$

$k \geq 1$. $S_k(\Gamma_1(N)) \circlearrowleft$ T_ℓ, ℓ prime
$\langle a \rangle$, $(a, N) = 1$

f eigenform $\longrightarrow (a_\ell)_{\ell \text{ prime}}$, $\varepsilon(a)$.

$f | T_\ell = a_\ell \cdot f$

$f | \langle a \rangle = \varepsilon(a) f$

$\varepsilon : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}$

<u>Recall</u> $\exists$ E number field

$\forall \ell$, $a_\ell \in \mathcal{O}_E$, $\varepsilon(a)$ (E field of coeff.)

higher $k \geq 2$, proof by using sing. coh. of $Y_1(N) = \Gamma_1(N) \backslash \mathfrak{h}$.
coher.
$k = 1$ proof by integral structures on $Y_1(N)_{\mathbb{Z}}$.
Dow.

<u>Th</u> f eigenform $\rightsquigarrow$ cont. hom. ρ_f
p prime
$\overline{\mathbb{Q}} \overset{i_p}{\hookrightarrow} \overline{\mathbb{Q}}_p$
$\rho_{f,p} : G_{\mathbb{Q}} \longrightarrow GL_2(\overline{\mathbb{Q}}_p)$
1) unr. outside Np
2) $\mathrm{char}(\rho_{f,p}(Fr_\ell)) = X^2 - a_\ell X + \ell^{k-1} \varepsilon(\ell$

proof $\begin{cases} k \geq 2 & \ell\text{-adic coh.} \\ k = 1 & \text{approx.} \end{cases}$

Complements

1) $\det \rho_{f,p} = \chi^{1-k} \cdot \varepsilon^{gal}$

$\chi : G_{\mathbb{Q}} \to \mathbb{Z}_p^\times$, $(\zeta_{p^n})^\sigma = \zeta_{p^n}^{\chi(\sigma) \in \mathbb{Z}_p^\times}$

$\varepsilon^{gal}(\underset{\substack{\uparrow \\ \text{geom. Frob.}}}{Fr_\ell}) = \varepsilon(\ell)$

$\Rightarrow \det \rho_{f,p}(c) = -1$. "odd rep."

2) $\rho_{f,p}$ is irred. (cuspidality of f)

3) $\exists F$ p-adic field, $F \supset \mathbb{Q}_p(E)$
$\rho_{f,p}$ is defined over F.

F, $\mathcal{O}_F$, ϖ, $K = \mathcal{O}_K/(\varpi)$

choosing a stable $\mathcal{O}_F$-lattice, $\bar\rho = (\rho_{f,p} \bmod \varpi)^{ss}$ well-defined

Serre's Conj.: Let $\bar\rho : G_{\mathbb{Q}} \to GL_2(K)$ cont. — finite field of char p.

$\bar\rho$ $\left.\begin{matrix}\text{abs. irred.} \\ \text{odd}\end{matrix}\right\} \Rightarrow \exists f$ eigen $\in S_k(N)$, $\left(\begin{matrix} k \geq 2 \\ p \nmid N \end{matrix} \right.$

s.t. $\bar\rho_{f,p} = \bar\rho$

geometricity:

$$\rho: G_{\mathbb{Q}} \longrightarrow GL_n(F)$$ p-adic field

ρ is geometric if

$$\begin{array}{cc} v & \overline{\mathbb{Q}} \\ | & | \\ p & \mathbb{Q} \end{array}$$

1) ρ unr. outside a finite set of primes
2) $\rho|_{G_{\mathbb{Q}_p}}$ is potentially semi-stable (PST)

point: definition does not involve geometry

$G_{\mathbb{Q}_p} \hookrightarrow G_{\mathbb{Q}}$

$$PST \Rightarrow HT \text{ (Hodge-Tate)}$$

HT: $\dim_F V = n$.

$G_{\mathbb{Q}_p} \curvearrowright V$

$$\underbrace{V \otimes_F \mathbb{C}_p}_{\text{semil. } G_{\mathbb{Q}_p}\text{-repr.}} \overset{HT}{=} \bigoplus_{i=1}^{n} \mathbb{C}_p(-w_i)$$

$w_i \in \mathbb{Z}$

$(w_1, \ldots, w_n)$: HT-weights

(If $w_i \neq w_j \ \forall i \neq j$, regular HT weights
(If not, singular

Conj. (Fontaine-Mazur)

Any cont. hom. $\rho: G_{\mathbb{Q}} \to GL_2(F)$

abs. irred., geometric $\} \Rightarrow \exists f \in S_{k=w+1}(N)$

$a \in \mathbb{Z}$, $\rho \otimes \chi^{-a}$ has weights $0, w$, $0 \leq w$

$\rho_{f,p} = \rho \otimes \omega^{-a}$

For Tilouine, $H^1(E)$ has wt. 0, 1 (some people prefer negative)

Rk. $\rho_{f,p}$ are geometric.

$k \geq 2$, $\rho_{f,p}$ is a Galois summand in

$$\begin{array}{c} E \\ \pi \downarrow \\ Y_1(N) \end{array} \quad H^1_{\acute{e}t}(Y_1(N) \times \overline{\mathbb{Q}}_\ell, \mathrm{Sym}^{k-2} R^1\pi_* \mathbb{Z}_p) \otimes F$$

$k=1$. No embedding in étale coh. of mod. curves.
Need that this repr. is Artin. —

$$S_1(N) = H^0(Y_1(N), \underset{\substack{| \\ \text{line bundle}}}{\underline{\omega}_1}) \not\subset H^1(Y_1(N), \ldots)$$

2) Modularity Conj. for $GSp(4)$

$$\mathfrak{h}_2 = \{ Z = {}^tZ \in M_2(\mathbb{C}) : \mathrm{Im}(Z) > 0 \}$$

$Sp_4(\mathbb{R}) \curvearrowright \mathfrak{h}_2$ $\quad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, $\quad gZ = (AZ+B)(CZ+D)^{-1}$

$$j(g, Z) = CZ + D \in GL_2(\mathbb{C})$$

$$K = (k_1, k_2), \quad k_1 \geq k_2 \geq 0$$

$$GL_2(\mathbb{C}) \overset{\rho_K}{\curvearrowright} W_K = \mathrm{Sym}^{k_1-k_2} \otimes \det{}^{k_2}(\mathbb{C}^2)$$

$$\dim W_K = 1 \iff k_1 = k_2$$

$$\Gamma(N) = \{ \gamma \in Sp_4(\mathbb{Z}) : \gamma \equiv 1_4 \mod N \}$$

$$\Gamma(N) \subset \Gamma \subset Sp_4(\mathbb{Z})$$

e.g. $\gamma \equiv \begin{pmatrix} 1 & \\ 0 & * \\ 0 & \\ 0 & \end{pmatrix}$ mod N, etc ...

$M_k(\Gamma)$, $S_k(\Gamma) = \{ f : \mathfrak{h}_2 \to W_k(\mathbb{C})$ hol. autom. (cuspidal if for $S_k(\Gamma)$) $\}$

$f|_k g$, $g \in Sp_4(\mathbb{R})$, $g \in GSp_4^+(\mathbb{R})$

$\left(\text{we use } \begin{pmatrix} & 1_2 \\ -1_2 & \end{pmatrix} \text{ instead of } \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix} \right.$ → Klingen?

the latter might be better for repr. theory purposes, but we use the former for the purpose of modular forms)

$$f|_k g = (\det g)^{k/2} \, J(g, z)^{-k} f(gz)$$

For autom. in $M_k(\Gamma)$, $S_k(\Gamma)$, we need:

$$f|_k \gamma = f \quad \forall \gamma \in \Gamma$$

For cuspidality, we need:

$\forall \gamma \in Sp_4(\mathbb{Q})$, $\lim_{t \to \infty} f|_k \gamma \begin{pmatrix} it & 0 \\ 0 & z \end{pmatrix} = 0$, $\operatorname{Im} z > 0$ (Siegel's Φ-operator)

<u>Def</u>: k is <u>higher</u> or <u>cohomological</u> if $k_1 \geq k_2 \geq 3$. Otherwise it is <u>low</u>

$Y_\Gamma = \Gamma \backslash \mathfrak{h}_2$ — 3-dim. complex variety

s.t. $S_k(\Gamma) \hookrightarrow H^3(Y_\Gamma, V_{\underline{a}}(\mathbb{C}))$ → repr of highest wt. $a = (a_1, a_2)$

Eichler-Shimura map

$k_1 = a_1 + 3$

$k_2 = a_2 + 3$

$a_1 \geq a_2 \geq 0$

(see next page for ref)

<u>Ref</u>. Hida, J. Inst. Math Jussieu. 1, 2002, sec. 3.8.

Hecke operators:

ℓ prime, $\ell \nmid N$

$$T_{\ell,1} = \Gamma \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ell & \\ & & & \ell \end{pmatrix} \Gamma$$

$$T_{\ell,2} = \Gamma \begin{pmatrix} 1 & & & \\ & \ell & & \\ & & \ell^2 & \\ & & & \ell \end{pmatrix} \Gamma$$

$$T_{\ell,0} = \Gamma \begin{pmatrix} \ell & & & \\ & \ell & & \\ & & \ell & \\ & & & \ell \end{pmatrix} \Gamma$$

acts on $S_k(\Gamma)$, $(Y_\Gamma, V_a(\mathbb{Z}))$ as corresp. ω is Hecke invariant

$\sigma_\ell \in Sp_4(\mathbb{Z})$, $\sigma_\ell \equiv \begin{pmatrix} \ell^{-1} & & & \\ & \ell^{-1} & & \\ & & \ell & \\ & & & \ell \end{pmatrix} \ (N)$

(<u>Ref.</u> Andrianov, Quadratic Forms & Hecke rings Sec. 3.1.5 & Chap. 3.3

R. Taylor, Duke 1992)

(For GL_2: $T_{\ell,1} = \Gamma_1(N) \begin{pmatrix} 1 & \\ & \ell \end{pmatrix} \Gamma_1(N)$

$T_{\ell,0} = \Gamma_1(N) \begin{pmatrix} \ell & \\ & \ell \end{pmatrix} \sigma_\ell \, \Gamma_1(N)$

$f|_{T_{\ell,0}} = \ell^{k-2} \varepsilon(\ell) f$)

$\mathcal{H}^N = \mathbb{Z}[T_{\ell,0}, T_{\ell,1}, T_{\ell,2}]_{\ell \nmid N, \text{ prime}}$

acts on $S_k(\Gamma)$ & $H^3(Y_\Gamma, V_{\underline{a}})$ compatibly

<u>Prop:</u> $f \in S_k(\Gamma)$ eigenform, $k = (k_1, k_2)$

$\exists$ E number field, $\forall \ell \nmid N$ prime. $a_{\ell,i} \in \mathcal{O}_E$.

because:

If k is coh.

$$H^3(Y_\Gamma, V_{\underline{a}}(\mathbb{C})) = \underbrace{H^3(Y_\Gamma, V_a(\mathbb{Z}))}_{\text{f.g.}} \otimes \mathbb{C}$$

If $k_1 \geq k_2$, is low, then

$$S_k(\Gamma) = H^0(Y_\Gamma, \underset{\text{vector bundle}}{\underline{\omega}^{\otimes k}}) \quad (\not\subset H^3(Y_\Gamma \ldots))$$

|| using integral structure

$$S_k(\Gamma, \mathbb{Z}) \otimes \mathbb{C}$$

<u>Def</u>. A Hecke eigensystem ass. to an eigenform f is the character

$$\theta_f : \mathcal{H}^N \longrightarrow \mathbb{C}$$

$$T \longmapsto \theta_f(T)$$

$$f|_T = \theta_f(T) \cdot f$$

For k c-hom.

If $f \in S_k(\Gamma)$, then θ_f occurs
is eigen.
in $H^3(Y_\Gamma, V_{\underline{a}}(\mathbb{C}))$

From the first lecture =

GL_2

$\boxed{\text{For } k \geq 2}$

$$(1) \qquad S_k(\Gamma) \xhookrightarrow{\omega} H^1(Y_1(N), V_{k-2}(\mathbb{C}))$$

$V_{k-2}(\mathbb{C}) = \mathrm{Sym}^{k-2}(\mathbb{C}^2)$ as representation of $SL_2(\mathbb{Q})$ → get local system

$$f \longmapsto \left[f(z) \begin{pmatrix} z \\ 1 \end{pmatrix}^n dz \right]$$

de Rham cohomology class, $n = k-2$

$\begin{pmatrix} z \\ 1 \end{pmatrix}^n \in \mathrm{Sym}^n(\mathbb{C}^2)$

ω = Eichler - Shimura map ; it is Hecke - equivariant

$$(1') \qquad \rho_{f,p} \subset H^1_{\acute{e}t}(Y_1(N) \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \underbrace{V_{k-2}(\mathbb{Q}_p)}) \otimes F$$

$\rho_{f,p}$: Galois summand

$V_{k-2}(\mathbb{Q}_p)$: smooth étale sheaf associated to $\mathrm{Sym}^{k-2}(\mathbb{Q}_p^2)$

Hence the terminology "k cohomological".

Theorem: (Faltings): $(1') \Longrightarrow$ HT weights of $\rho_{f,p}$ are $0, k-1$ (regular!)

$\boxed{k=1}$: $\quad S_1(\Gamma) = H^0(Y_1(N)^{an}, \underline{\omega}^{an})$

line bundle

So, it is given by cohomology,

but coherent cohomology, not de Rham (or not singular)

Theorem: (Deligne Serre) Nevertheless, $\rho_{f,p}$ is Artin hence HT, with weights $0, 0$ (singular!)

G Sp_4 $\qquad \kappa = (k_1, k_2)$

$\boxed{\text{For } k_1 \geq k_2 \geq 3}$

$$(2) \qquad S_\kappa(\Gamma) \xhookrightarrow{\omega} H^3(Y_\Gamma, V_a(\mathbb{C}))$$

canonical

$\exists$ canonical

with $\underline{a} = (a_1, a_2)$ $\quad a_i = k_i - 3$; $a_1 \geq a_2 \geq 0$

$V_{\underline{a}}(\mathbb{Q})$: representation of highest weight $\underline{a}$ of $Sp_4(\mathbb{Q})$.

$V_{\underline{a}}(\mathbb{C})$ local system in $\mathbb{C}$-vect. spaces associated to $V_{\underline{a}}(\mathbb{Q})$

<u>Definition of ω</u> :

$\exists!$ polynomial function $p_{\underline{a}} : \mathfrak{h}_2 \to \mathrm{Hom}(W_{\underline{a}}, V_{\underline{a}})$ (up to scalar) such that

$\forall g \in Sp_4(\mathbb{R})^{*_0}$, $\quad p_{\underline{a}}(g(Z)) \circ \rho_{\underline{a}}(j(g,Z)) = g \circ p_{\underline{a}}(Z)$

then

$$\omega(f) = \left[p_{\underline{a}}(Z)\,(f(Z)) .\, dz_{11} \wedge dz_{12} \wedge dz_{22} \right]$$

example: $\underline{a} = (1,0)$ $\quad p_{\underline{a}}(Z) = \begin{pmatrix} Z \\ 1_2 \end{pmatrix} \in \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^4)$

(it makes sense because $W_k = W_{\underline{a}} \otimes \det^3$)

$[\quad] =$ de Rham cohom. class of this holomorphic 3-form.

ω is Hecke-equivariant.

We'll see if f is not "particular" and $f \in S_k(\Gamma)$ eigen,

then (2') $\quad \rho_{f,p} \underset{\text{Galois summand}}{\subset} H^3_{et}(Y_\Gamma \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \underbrace{V_{\underline{a}}(\mathbb{Q}_p)}) \otimes F$

smooth étale sheaf attached to the represent. $V_{\underline{a}}(\mathbb{Q})$ of $Sp_4(\mathbb{Q})$.

Hence the terminology :
"k is cohomological".