Khare's lecture on the Serre's conjecture

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1. Introduction

Khare will present his recent joint work with Wintenberger on the odd conductor case of Serre's conjecture.

Recall the statement of the conjecture: let

$$\bar{\rho} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F})$$

be an odd, absolutely irreducible mod p representation, with $\mathbb{F}$ a finite field of characteristic p. Denote by $N(\bar{\rho})$ and $k(\bar{\rho})$ the conductor and weight respectively of $\bar{\rho}$ defined as in [**4**]. There, Serre conjectured that $\bar{\rho}$ arises from a newform of level $N(\bar{\rho})$ and weight $k(\bar{\rho})$.

The approach of Khare and Wintenberger is via "lifting theorems". These can be divided into two types: Galois-theoretic liftings and modularity liftings.

On the Galois-theoretic side, one tries to prove that such a $\bar{\rho}$ arises as reduction mod p of a p-adic Galois representation ρ, which is minimally ramified, in the sense that ρ has the same conductor as $\bar{\rho}$ outside p, and is crystalline at p with Hodge-Tate weights $(0, k(\bar{\rho}) - 1)$. This would follow if one can show that the minimal deformation ring for $\bar{\rho}$, $\mathcal{R}(\bar{\rho})$, has no p-torsion. Works of Fujiwara, Skinner-Wiles, Taylor play an important role here.

On the other hand, one tries to prove modularity lifting theorems: assuming $\bar{\rho}$ is modular, and that ρ is a lift of $\bar{\rho}$, show that ρ is itself modular. These theorems is the work of many people, Wiles, Taylor, Breuil, Conrad, Diamond, Fujiwara, Kisin, Savitt, Skinner *et al.*

With enough lifting theorems of these two types, the strategy of proving Serre's conjecture (at least in the conductor one case) is very roughly speaking an induction on the residue characteristic p: given $\bar{\rho}$ as above, it's lifted to a p-adic representation ρ; using Taylor's potential modularity theorems, one can propagate ρ to a compatible system (ρ_λ), with $\rho_p = \rho$. By induction hypothesis, ρ_l mod l is modular for $l < p$ (the starting point of the induction would use the results of Tate and Serre). One tries to find such an l for which the modularity lifting theorems can be applies to ρ_l mod l. Then the modularity of ρ_l implies that of ρ_p, hence of $\bar{\rho}$.

Finally we note that results on Serre's conjecture, when combined with the modularity theorems, allows one to prove more cases of the Fontaine-Mazur conjecture.

References

[1] C.Khare, *On Serre's modularity conjecture*, preprint.

[2] C.Khare, *On Serre's modularity conjecture for* 2*-dimensional mod* p *representations of* $\mathrm{Gal}(\bar{\mathbb{Q}}, \mathbb{Q})$ *unramified outside* p, preprint.

[3] C.Khare, J.P.Wintenberger, *Serre's modularity conjecture: the odd conductor case (I)*, preprint.

[4] J.P.Serre, *Sur les représentations modulaires de degré de* $\mathrm{Gal}(\bar{\mathbb{Q}}, \mathbb{Q})$, Duke Math J. p.179-230, 1987.