Motivic sheaves over a curve

This joint work with A. Beilinson and A. Otwinowska is the first step in our attempt to develop a theory of motivic sheaves over any base scheme. For any scheme S, we expect to have a DG category $\mathcal{M}(S)$ such that if S is the spectrum of a field $\mathcal{M}(S)$ coincides with Voevodsky's category and such that, for any closed subscheme $D \overset{i}{\hookrightarrow} S \overset{j}{\hookleftarrow} S - D = U$, there are functors:

$$\mathcal{M}(D) \xrightarrow{i_*} \mathcal{M}(S) \xrightarrow{j^*} \mathcal{M}(U) \tag{1}$$

which identify $\mathcal{M}(D)$ with a full subcategory of $\mathcal{M}(S)$ and $\mathcal{M}(U)$ with the DG quotient of $\mathcal{M}(S)$.

Assume, for the moment, that we know the categories $\mathcal{M}(D)$ and $\mathcal{M}(U)$. We shall see that all the extensions (1) of $\mathcal{M}(U)$ by $\mathcal{M}(D)$ (viewed as abstract DG categories) with the property that the subcategory $Hot(\mathcal{M}(D)) \xrightarrow{i_*} Hot(\mathcal{M}(S))$ (where $Hot(?)$ denotes the associated triangulated category) is both left and right admissible are parametrized by functors $\Psi : \mathcal{M}(U) \to \mathcal{M}(D)$.

Assume now that the base scheme S is a curve C over a field k. Ignoring at the moment that the set of all closed point is not a closed subscheme of C it is clear from the previous paragraph that to define the category of motivic sheaves over C all we need is to specify the functors $\mathcal{M}(\eta) \to \mathcal{M}(k(a))$, where η is the generic point of C and $a \in C$ a closed point. If the motive over η is represented by a scheme X_η over η its image under the above functor should be viewed as the motivic punctured tubular neighborhood $PTN_a(X_\eta)$ of X_η at a. The topological picture suggests that if $k(a) = k$ and the scheme X_η is proper over η then the motivic punctured neighborhood $PTN_a(X_\eta)$ should be quasi-isomorphic to to the following complex of schemes over k:

$$PTN_a(X_\eta) \simeq cone(X_a \coprod X_U \to X)[-1],$$

where X is any proper scheme over C whose fiber over the generic point is X_η and U denotes the complement to a in C .

I will explain how to construct a functor $PTN_a : \mathcal{M}(\eta) \to \mathcal{M}(k(a))$ with the above property and which commutes with the etale realization.